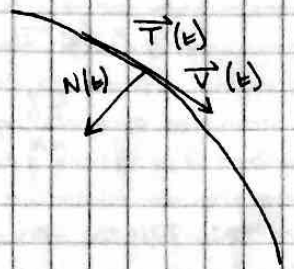
1/21 LECTURE

$$T(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$



DEFINE: $\vec{N}(t) = \frac{1}{K} \frac{d\vec{T}}{ds}$ UNIT NORMAL VECTOR

$$K = \left| \frac{d\vec{T}}{ds} \right|$$

$$\frac{dT}{ds} = \frac{dT}{dt} \frac{dt}{ds} = \frac{d\vec{T}}{dt} \frac{1}{|\vec{v}(t)|}$$

$$\frac{ds}{dt} = |\vec{v}(t)|$$

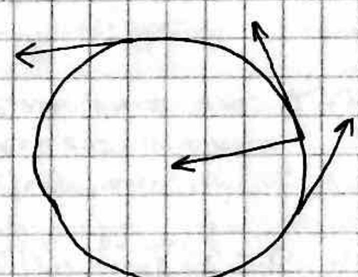
$$K = \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{|\vec{v}(t)|} \left| \frac{d\vec{T}}{dt} \right|$$

Ex. $\vec{r}(t) = 2\langle \cos t, \sin t \rangle$

$\vec{v}(t) = 2\langle -\sin t, \cos t \rangle$

$|\vec{v}(t)| = 2$

$$s(t) = \int_0^t |\vec{v}(u)|\, du = \int_0^t 2\, du = 2t$$



$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \langle -\sin t, \cos t \rangle$$

TO COMPUTE K, COMPUTE:

$$\frac{d\vec{T}}{dt} = \frac{d}{dt}\langle -\sin t, \cos t \rangle = \langle -\cos t, -\sin t \rangle$$

$$K = \frac{1}{|\vec{v}(t)|} \left| \frac{d\vec{T}}{dt} \right| = \frac{1}{2}\sqrt{(-\cos t)^2 + (-\sin t)^2} = \frac{1}{2}$$

IN GENERAL, FOR MOTION ALONG A CIRCLE OF RADIUS R, ONE CAN PROVE THAT

$$K = \frac{1}{R}$$

1/21 LECTURE:

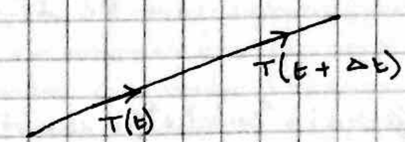
EX. STEADY MOTION ALONG A LINE $\vec{r}(t) = \vec{r}_0 + t\vec{v}$; WITH $\vec{v} = \langle a, b, c \rangle$

CONSTANT

$$\vec{v}(t) = \vec{r}\,'(t) \equiv \vec{v};\ |\vec{v}(t)| = |\vec{v}|$$

$$\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|}$$

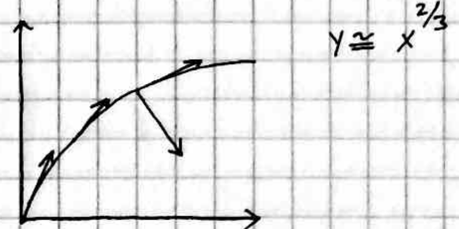
$$\frac{d\vec{T}}{dt} = 0$$



WE CANNOT DEFINE $\vec{N}(t)$ UNAMBIGUOUSLY

EX.

$$\vec{r}(t) = \left\langle \frac{t^3}{3}, \frac{t^2}{2} \right\rangle$$



$$\vec{v}(t) = \langle t^2, t \rangle = t\langle t, 1 \rangle$$

$$|\vec{v}(t)| = t\sqrt{t^2+1}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{t\langle t, 1\rangle}{t\sqrt{t^2+1}} = \frac{\langle t, 1 \rangle}{\sqrt{t^2+1}}$$

$$\boxed{\vec{N} \perp \vec{T} \text{ so, } \vec{N} \cdot \vec{T} = 0}$$

$$\frac{d\vec{T}}{dt} = \ldots$$

$\vec{N}$ MUST BE $\perp$ TO $\vec{T}$, SO WE GUESS

$$\boxed{\vec{N} = \frac{\langle 1, -t \rangle}{\sqrt{1+t^2}}}$$

1/24

LECTURE:

DEMONSTRATE THAT $\vec{T} \cdot \vec{N} = 0$ (ASSUME $\vec{T}, \vec{N}$ ARE DEFINED) THIS IS A MORE GENERAL STATEMENT THAT IF $\vec{w}(t)$, $\frac{d}{dt}\vec{w}(t)$, ARE WELL DEFINED, THEN $\vec{w}(t) \cdot \frac{d}{dt}\vec{w}(t) = 0$ IF AND ONLY IF $|\vec{w}|$ = CONSTANT.

$\vec{w}(t) \cdot \frac{d}{dt}\vec{w}(t) = 0$ IF/ONLY IF $|\vec{w}(t)|$ = CONSTANT

$\vec{w}(t) = \langle w_1(t), w_2(t), w_3(t) \rangle \qquad \frac{d}{dt}\vec{w}(t) = \langle w_1'(t), w_2'(t), w_3'(t) \rangle$

$$\vec{w} \cdot \frac{d}{dt}\vec{w} = (w_1)w_1' + (w_2)w_2' + (w_3)w_3'$$

$$= \frac{1}{2}\left[w_1^2 + w_2^2 + w_2^3\right]' = 0$$

HELP SESSION BEXL 323

$$w_1^2 + w_2^2 + w_3^2 = \text{CONSTANT}$$

$$|w|^2 = \text{CONSTANT}$$

1/26

LECTURE

TEST FOCUSED ON CHAPTER 11

REVIEW PROBLEMS PG. 771

13, 18, 21, 22, 32, 38

(13) $S = \left\{ (x,y,z) : (x-1)^2 + (y-0)^2 + (z-(-1))^2 = 4^2 \right\}$

(18) $v = -3j + 4k \,,\; v = -4i + j + 5k$

$= \langle 0,-3,4 \rangle \,,\; \langle -4,1,5 \rangle$

$\vec{u} \cdot \vec{v} = 17$

CONTINUE TO FIND θ

(21) $\langle 2,-6,9 \rangle, \langle -1,0,6 \rangle$

$u \times v$

$$\begin{vmatrix} 2 & -6 & 9 \\ -1 & 0 & 6 \end{vmatrix} = (-36-0)i - (12+9)j + (6)j$$

$-36i + 3j + 6k$

$\vec{w} \quad \langle -36, \overset{-21}{\cancel{3}}, -6 \rangle$

$\vec{w} \perp \vec{u}, \; \vec{w} \perp \vec{v}$

ARCLENGTH:

$$L = \int |\vec{v}(t)| \, dt$$

(22) $\langle 2,0-2 \rangle, \langle 2,2,0 \rangle$

SAME AS ABOVE

(32) $r = 3 + 2\cos\theta$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$\vec{r}(t) = \langle f(t)\cos t, f(t)\sin t \rangle$

$\vec{v}(t) = \langle f'(t)\cos t, f'(t)\sin t \rangle + f\langle -\sin t, \cos t \rangle$

$|\vec{v}(t)| = \sqrt{(f)^2 + (f'(t))^2}$

$$L = 2\int_0^{\pi} \sqrt{13 + 12\cos t} \, dt$$

2/1

LECTURE

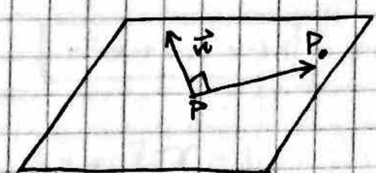
$f : \mathbb{R}^3 \to \mathbb{R}$

EQUATION OF A PLANE

$P_0 = (x_0, y_0, z_0)$, $\vec{n} = \langle a, b, c \rangle$

A PLANE THAT PASSES THROUGH P_0 AND HAS A NORMAL VECTOR $\vec{n}$ HAS THE EQUATION:

$\vec{n} \cdot PP_0 = 0$ WHERE $P = (x, y, z)$



$\langle a, b, c \rangle \cdot \langle x - x_0, y - x_0, z - z_0 \rangle = 0 \Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\boxed{ax + by + cz = d, \text{ WHERE } d = ax_0 + by_0 + cz_0}$ CANONICAL FORM

EX. FIND AN EQUATION OF A PLANE THAT PASSES THROUGH $(-1, 2, \sqrt{\pi})$ AND IS NORMAL TO $\langle -e^e, 0, \sin 1/3 \rangle = \vec{n}$

$\langle -e^e, 0, \sin 1/3 \rangle \cdot \langle x+1, y-2, z-\sqrt{\pi} \rangle$

$-e^e(x+1) + 0(y-2) + \sin 1/3 (z - \sqrt{\pi})$

$-e^e x + \sin 1/3 = e^e + \sqrt{3} \sin 1/3$

CHECK WHETHER $\vec{0}$ IS IN THE PLANE $\Rightarrow$ DOES L = R @ 0?

GIVEN $z = -2x + 4y + 3$, FIND PLANE PARALLEL TO THIS PLANE PASSING THROUGH $\vec{0}$. FIND VECTOR NORMAL TO PLANE

1. WRITE IN FORM $\vec{n} \cdot \vec{PB} - 0$

$-2x + 4y - z + 3 = 0$

$-2x + 4y - z = -3$

$\langle -2, 4, -1 \rangle \cdot \langle x, y, z \rangle = d$, $\vec{n}$ = a NORMAL VECTOR TO THE PLANE

WE CAN USE THE $\vec{n} = \langle 2, -4, 1 \rangle$ AS THE NORMAL VECTOR TO ANY PARALLEL PLANE

$2x + (-4)y + z = 0$

2/1

LECTURE

EX. FIND AN EQUATION OF THE PLANE THAT PASSES THROUGH:

$P_1(1,2,3)$, $P_2(-1,2,3)$, $P_3(-3,-2,2)$

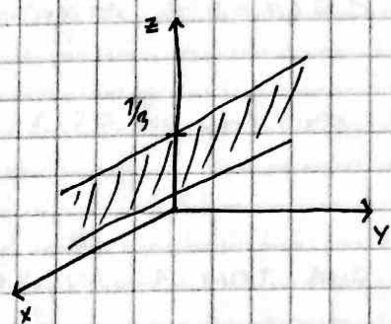
$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$

$$\begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \left| \begin{array}{ccc} -2 & 0 & 0 \\ -6 & -3 & -1 \end{array} \right| \end{array} = \langle 0, -2, 6 \rangle$$

$\langle 0,-2,6 \rangle \cdot \langle x-1,\ y-2,\ z-3 \rangle = 0$

$\boxed{-2y + 6z = 14}$

$z = \dfrac{7+y}{3}$



2/3

LECTURE:

READ SECTION 12.4 ON PARTIAL DERIVATIVES.

12.3 LIMITS AND CONTINUITY, $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$

RECALL

$$\lim_{x \to a} f(x) = L$$

DEFINITION:

$f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, f IS CONTINUOUS AT $(a,b) \subseteq D$

$\lim_{(x,y) \to (a,b)} f(x,y) = L$ for $\epsilon > 0$ WE CAN FIND $\delta > 0$: $\Rightarrow |f(x,y) - L| < \epsilon$

KNOWN FACTS:

1. $f(x,y) \equiv C$, WITH SOME $C \in R$ HAS A LIMIT AND IS CONTINUOUS AT ANY $(x,y) \in \mathbb{R}^2$

2. IF (F, G) ARE CONTINUOUS:

$f \pm G \to$ CONT.
$f \cdot G \to$ CONT
$f / G \to$ CONT, AS LONG AS $G \neq 0$

3. $f(x,y) = x$ IS CONT.
$G(x,y) = y$ IS CONT.

$(1,2,3) \Rightarrow$ POLYNOMIALS IN (x,y) ARE CONT.
$\Rightarrow$ RATIONAL FUNCTIONS OF (x,y) IF DENOM IS $\neq 0$

TRIG / EXP. / LOG FUNCTIONS. / ROOTS ARE ALL CONTINUOUS.

EX. $f(x,y) = 2x^2 - 3xy + \sin(x)y^2$

$$\lim_{(x,y) \to (1,-1)} f(x,y) = \lim_{(x,y) \to (1,-1)} \underset{2}{2} \cdot \underset{(1)}{x} \cdot \underset{(1)}{x} - \underset{3}{3} \cdot \underset{(1)}{x} \cdot \underset{(-1)}{y} + \underset{\sin(1)}{\sin(x)} \cdot \underset{(-1)}{y} \cdot \underset{(-1)}{y}$$

$$2 + 3 + \sin(1) = 5\sin(\cdot)$$

4. IF $f(x,y)$ HAS A LIMIT AT (a,b) AND YOU HAVE A FUNCTION $g(x,y)$ THAT HAS A $\lim_{(x,y) \to (x_0, y_0)} f(g(x_0, x_1), g_2(x_0, y_0))$

2/3

LECTURE

Ex. $f(x,y) = \dfrac{1}{x^2+y^2}$

$$\lim_{(x,y)\to(1,-1)} f(x,y) = \frac{1}{(1)^2+(-1)^2} = \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} \Rightarrow \text{DNE}$$

Ex. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x^2-y^2}$

2/7

LECTURE

12.4 DIFFERENTIATION OF $\mathbb{R}^2 \to \mathbb{R}$; $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2} ; & (x,y) \neq \vec{0} \\ 1 & (x,y) = 1 \end{cases}$$

FUNCTION DOES NOT HAVE A LIMIT

$f(x,y) = 5 - x^2 - 10y^2$

$f : D \in \mathbb{R}^2 \to \mathbb{R}$, $(a,b) \in D$

PARTIAL DERIVATIVE:

$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial f}{\partial y} = f_y$

$$\frac{\partial f}{\partial x}(a,b) = \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \frac{f(b+h,a) - f(a,b)}{h}$$

DEF EX.

$$f(x,y) = x^2 \qquad \frac{\partial f}{\partial x}(a,b) = \frac{f(a+h,b) - f(a,b)}{h} = \lim_{H \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{H \to 0} \frac{2ha + h^2}{h}$$

$$= \lim_{H \to 0} (2a + h) = 2a$$

$f(x,y) = 5 - x^2 - 10y^2$

$\frac{\partial f}{\partial x} = -2x$

$\frac{\partial f}{\partial y} = -20y$

IN SINGLE VARIABLE CALCULUS

$f(x) \approx f(a) + f'(a)(x-a)$ → TANGENT LINE APPROXIMATION

DEF: f IS DIFFERENTIABLE AT (a,b) IF A LOCAL LINEAR APPROXIMATION CAN BE FOUND

2/9

LECTURE

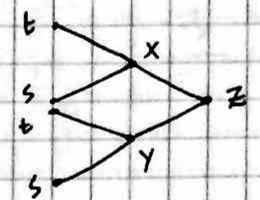
12.5 CHAIN RULE

EX. $z = y\sin x$

$\frac{\partial z}{\partial x} = y\cos x$

$\frac{\partial z}{\partial y} = \sin x$

EX.



$z = y\sin x$

$x = s^2 + t^2$

$y = st$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$

$\frac{\partial x}{\partial t} = 2t \qquad \frac{\partial x}{\partial s} = 2s$

$\frac{\partial y}{\partial t} = s \qquad \frac{\partial y}{\partial s} = t$

$\frac{\partial z}{\partial t} = (y\cos x)2t + (\sin x)s = 2st^2\cos(s^2+t^2) + s\sin(s^2+t^2)$

IMPLICIT DIFFERENTIATION

ELLIPSE: $x^2 + 4y^2 = 1$

$\frac{d}{dx}\left(x^2 + 4y^2\right) = 0$

$2x + 8y\frac{dy}{dx} = 0$

$8y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-1x}{4y}$

$\frac{dy}{dx} = \frac{-x}{4y}$ OKAY WITH $y \neq 0$

2/9

LECTURE

$F(x,y) = 0 \Rightarrow$ GET $\frac{dy}{dx}$

$$\frac{d}{dx}\left[f(x,x)\right] = \frac{d}{dx} 0$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{-F_y}$$

WORKS IF F IS DIFFERENTIABLE AT (X_0, Y_0) AND IF $F_y \neq 0$

EX.

$$ye^{xy} - 2 = 0$$

$$\frac{d}{dx}\left[ye^{xy} - 2\right] = 0$$

$$\frac{dy}{dx} e^{xy} + y\frac{d}{dx} e^{xy} = 0$$

$$\frac{dy}{dx} e^{xy} + ye^{xy}\frac{d}{dx}(x,y) = 0$$

$$\frac{dy}{dx} e^{xy} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

SOLVE FOR WHAT YOU WANT

2/11

HELP SESSION IN KIDDER 364. STARTS 6.

12.1-12.7 ON EXAM

SECTION 12.6

NOTE, IF f_{xy}, AND f_{yx} ARE CONTINUOUS THEN THEY ARE EQUAL

$$f_{xy} = f_{yx}$$

2/11

LECTURE

GRADIENT:

$$\nabla f \stackrel{\text{def}}{=} \langle f_x, f_y \rangle$$

DIRECTIONAL DERIVATIVE:

$D_{\vec{v}} f = \nabla f \cdot \vec{u}$ REQUIRED $|\vec{u}| = 1$

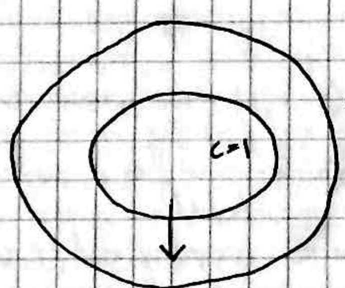
$D_{\vec{v}} f = |\nabla f| \cos\theta$

THE LARGEST CHANGE OF $f(\cdot)$ IS IN THE DIRECTION OF THE GRADIENT
'OR'
THE DIRECTION OF ∇f IS THE DIRECTION OF STEEPEST INCREASE

LET $\vec{u} = \langle u_1, u_2 \rangle$ SO THAT $|\vec{u}| = 1$

THEN $D_{\vec{u}} f = \lim_{h \to 0} \frac{f(a+u_1h, b+u_2h) - f(a,b)}{h}$

EX. $f(x,y) = x^2 + 5y^2 \rightarrow$ LEVEL CURVES ARE ELLIPSES



$\vec{u} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\nabla f = \langle 2x, 10y \rangle$

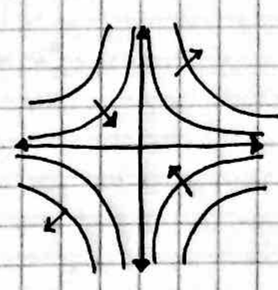
$D_{\vec{v}} f = \langle 2x, 10y \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$D_{\vec{v}} f = \frac{1}{\sqrt{2}} \langle -2x, 10y \rangle$

GRADIENT POINTS IN THE DIRECTION WHERE THE FUNCTION INCREASES.

GRADIENT IS ORTHAGONAL TO THE LEVEL CURVES

EX. $f(x,y) = xy \qquad y = \frac{c}{x}$



2/14

12.7

SURFACE IN $\mathbb{R}^3$ AND A TANGENT PLANE TO THAT SURFACE

GENERAL EQUATION OF A SURFACE $F(x,y,z)=0$

TWO THINGS TO DEFINE A PLANE:
- NORMAL VECTOR
- POINT

∇ GRADIENT CAN BE USED AS $\vec{n}$

EX.
$$F(x,y,z) = e^{(x^2+3y^2)z} - 1$$

FIND EQ OF TANGENT PLANE AT $(0,0,\pi)$

ARBIDARY z VALUE

$$\vec{n} = \nabla F\Big|_{(0,0,\pi)}$$

$$F_x = 2xze^{(x^2+3y^2)z}\Big|_{(0,0,\pi)} = 0$$

$$F_y = 6yze^{(x^2+3y^2)z} = 0$$

$$F_z = x^2+3y^2 e^{(x^2+3y^2)z} = 0$$

CANNOT FIND TANGENT PLANE AT $(0,0,0)$

EVALUATED AT $(1,1,0)$

$$\vec{n} = \nabla F\Big|_{(1,1,0)} = \langle 0,0,4\rangle$$

$$(x-1)\cdot 0 + (y-1)\cdot 0 + (z-0)\cdot 4 = 0$$

$4z=0$

$z=0$ PLANE IS THE (x,y) PLANE

EX. $z = x^2+3y^2 \Leftrightarrow z - x^2 - 3y^2 = 0$

FIND EQ. OF TANGENT PLANE AT $(0, -1/\sqrt{3}, 1)$

$$\nabla F = \langle -2x, -6y, 1\rangle_{(0,-1/\sqrt{3},1)} = \langle 0, \tfrac{6}{\sqrt{3}}, 1\rangle = \vec{n}$$

$$(x-0)\cdot 0 + \left(y+\tfrac{1}{\sqrt{3}}\right)\tfrac{6}{\sqrt{3}} + (z-1)\cdot 1 = 0$$

$$z = -1 - 2\sqrt{3}\,y$$

2/14

USE TANGENT PLANE FOR LOCAL LINEAR APPROXIMATION

$$f(x,y) = f(a,b) + f_x\Big|_{(a,b)}(x-a) + f_y\Big|_{(a,b)}(y-b)$$

DIFFERENTIAL

$f: \mathbb{R} \to \mathbb{R}$: $\Delta y \approx f'(a)\,\Delta x$

$dy = f'(a)\,dx$

$\frac{dy}{dx} = f'$

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\Delta z = f_x \Delta x + f_y \Delta y$$

$$dz = f_x\,dx + f_y\,dy$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

EX. $f(x,y) = \sqrt{x^2+y^2}$ AT $(3,-4)$

$f(3,4) = 5$ ESTIMATE $(3.06, -3.92)$

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle\Big|_{(3,-4)} = \left\langle 3/5, -4/5 \right\rangle$$

$$\Delta z = 3/5(3.06-3) + (-4/5)(0.08)$$

2/16 REVIEW:

(70) ESTIMATE THE CHANGE IN FUNCTION $f(x,y) = -2y^2 + 3x^2 + xy$ WHEN (x,y) CHANGES FROM $(1,-2)$ TO $(1.05,-1.9)$

$dz = f_x\,dx + f_y\,dy$

$\Delta x = 1.05 - 1$

$\Delta y = -1.9 - (-2) = 0.1$

$\vec{n} = \nabla F$

EVALUATE GRADIENT, SUB A, B, C IN TAGENT PLANE EQ.

(60.) FIND $\frac{dy}{dx}$, EVALUATE AT GIVEN POINT OF (a,b), FIND TANGENT LINE, GRADIENT OF TANGENT LINE, CHECK TO SEE IF THEIR ORTHO.

(54.)

DIRECTIONAL DERIVATIVE

$D_{\vec{u}}f = \nabla f \cdot \vec{u}$ ← MAKE SURE $\vec{u}$ IS A UNIT VECTOR

$\frac{1}{\sqrt{2}}(a - b)$

2/21 LECTURE

$f(x,y)$ $f: \mathbb{R}^2 \to \mathbb{R}$, ASSUME f IS SMOOTH

① $\nabla f_{(x_0,y_0)} = 0$

② SECOND DERIVATIVE TEST TO DETERMINE WHETHER AT (x_0, y_0) WE HAVE: LOCAL MIN, MAX, SADDLEPOINT.

HESSIAN

$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yy} & f_{yx} \end{bmatrix}$$

$D > 0$ LOCAL MIN/MAX

$D < 0$ SADDLE POINT

$D =$ INCONCLUSIVE

DISCRIMINANT

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yy} & f_{yx} \end{vmatrix}$$

EX. $f(x,y) = x^2 + y^2 + 4x - 2y$

$\nabla f = \langle 2x+4, 2y-2 \rangle = \vec{0}$

$$\begin{cases} 2x+4 \\ 2y-2 \end{cases} \Rightarrow \begin{matrix} x = -2 \\ y = 1 \end{matrix}$$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = \underbrace{4 > 0}_{\text{LOCAL MIN}}$$

2/23

LAGRANGE MULTIPLIERS

EX. FIND EXTREME VALUES OF $f(x,y) = x^2 + y^2$ SUBJECT TO $\underbrace{x+y-1}_{g(x,y)} = 0$

$$\begin{cases} x = t \\ y = 1-t \end{cases} \qquad F(t) = f(x(t), y(t)) = t^2 + (1-t)^2$$

$$0 = \frac{d}{dt} F(t) = \frac{d}{dt}\left(f(\vec{r}(t))\right) = \nabla f \cdot \vec{r}\,'(t)$$

FIND t: $\nabla f \perp \vec{T}(t)$

$\nabla f \parallel \vec{n}(t)$

$\downarrow$

$\nabla f \parallel \nabla g$

3. $\nabla f = \lambda \nabla g$ $\quad \lambda$ IS SCALAR MULTIPLE, LAGRANGE MULTIPLIER

SOLUTION

FIND x, y, λ

$\nabla f = \lambda \nabla g$ AND $g(x,y) = 0$

$\nabla f = \langle 2x, 2y \rangle$

$\nabla g = \langle 1, 1 \rangle$

$$\begin{matrix} 2x = \lambda \\ 2y = \lambda \\ x+y-1=0 \end{matrix} \Longrightarrow \begin{matrix} x = \frac{\lambda}{2} \\ y = \frac{\lambda}{2} \\ \frac{\lambda}{2} + \frac{\lambda}{2} = 1 \end{matrix} \qquad \begin{matrix} x = \frac{1}{2} \\ y = \frac{1}{2} \\ \lambda = 1 \end{matrix}$$

LAGRANGIAN

EX. $f(x,y) = x^2 + y^2 + 4x - 2y$ ON $R = \{(x,y) : x^2 + y^2 \le 16\}$

WE FOUND CRITICAL POINT $(-2, 1)$, MIN: $f(x,y) = -5$

FIND ABSOLUTE (GLOBAL) MAXIMUM ON R

PARAMATERIZE:

$x^2 + y^2 = 16$

$$\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$$

3. USE LAGRANGE MULTIPLIERS:

$\nabla f = \langle 2x+4, 2y-2 \rangle$

$\nabla G = \langle 2x, 2y \rangle$

$$\begin{cases} 2x + 4 = \lambda(2x) \\ 2y - 2 = \lambda(2y) \\ x^2 + y^2 = 16 \end{cases} \Rightarrow \begin{matrix} 2x(1-\lambda) = -4 \\ 2y(1-\lambda) = 2 \end{matrix} \qquad \begin{matrix} x = \dfrac{-2}{1-\lambda} \\ y = \dfrac{1}{1-\lambda} \end{matrix}$$

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$$\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 16$$

$$\frac{5}{(1-\lambda)^2} = 16$$

$$1-\lambda = \pm\frac{\sqrt{5}}{4}$$

$$x = \frac{-2}{\frac{\pm\sqrt{5}}{4}} \qquad y = \frac{1}{\frac{\pm\sqrt{5}}{4}}$$

$$x = \frac{\mp 8}{\sqrt{5}} \qquad y = \frac{\pm 4}{\sqrt{5}}$$

$$f\left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) \quad f\left(\frac{-8}{\sqrt{5}}, \frac{-4}{\sqrt{5}}\right)$$

2/25 MULTIPLE INTEGRATION

$$\iint_R f(x,y)\,dA \qquad R \subseteq R^2$$

AREA INTEGRALS

$$\iiint_V f(x,y,z)\,dV$$

VOLUME INTEGRALS

RIEMANN SUMS:

$$\iint f(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta x \Delta y$$

REGIONS TO BE RECTANGULAR

$$R = [a,b] \times [c,d] \qquad R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

FUBINI THEOREM

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

2/25

EX. $\iint (2x+3y+5)\,dA$ OVER $R = [-1,1] \times [0,3]$

$$\int_{-1}^{1}\left[\int_0^3 2x+3y+5\,dy\right]dx \Rightarrow \int dx\left[2xy + \frac{3y^2}{2} + 5y\Big|_{y=0}^{y=3}\right]$$

$$= \int_{-1}^{1} dx\left[\left(2xy + \frac{3y^2}{2} + 5y\right)_3 - \left[2xy + \frac{3y^2}{2} + 5y\right]_0\right] = \int_{-1}^{1} dx\left[\left(6x + \frac{27}{2} + 15\right) - 0\right]$$

$$\int_{-1}^{1}(6x + 28.5)\,dx \Rightarrow 3x^2 + 28.5x\Big|_{-1}^{1} \quad \cancel{31.5} = 57$$

EX.

$$\iint (x^2+xy)\,dA \qquad R = [1,2] \times [-1,1]$$

$$\int_{-1}^{1} dy \int_1^2 dx\,(x^2+xy) \Rightarrow \int_{-1}^{1} dy\left(\frac{x^3}{3} + \frac{x^2}{2}y\right)\Big|_1^2 \Rightarrow \frac{8}{3} + 2y - \left[\frac{1}{3} + \frac{y}{2}\right]$$

$$\int_{-1}^{1} dy\left[\frac{7}{3} + \frac{3}{2}y\right] = \frac{7}{3}y + \frac{3y^2}{4}\Big|_{-1}^{1} = \frac{7}{3}(1-(4)) + 0 = \frac{14}{3}$$

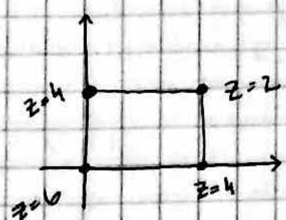
EX $$\int_0^2 dx \int_0^{\pi/2} dy\,(y^3\sin(xy^2)) \xrightarrow{\text{REORDER}} \int_0^{\pi/2} dy \int_0^2 dy\,(y^3\sin(xy^2)) = \int_0^{\pi/2} y^3\,dy\left[\int_0^2 dx\,\sin(xy^2)\right] =$$

$$\int_0^{\pi/2} y^3\,dy\left(\frac{-1}{y^2}\cos(xy^2)\right)\Big|_0^2 = -\int_0^{\pi/2} y\,dy\left[\cos(2y^2) - \cos(0\cdot y^2)\right] - \int_0^{\pi/2} y\left(\cos(2y^2) - 1\right]dy$$

$$\underbrace{-\int_0^{\pi/2} y\cos(2y^2)\,dy}_{\text{SUBSTITUTION}} + \int_0^{\pi/2} y\,dy \Rightarrow -\frac{1}{4}\sin(2y^2)\Big|_0^{\pi/2} + \frac{y^2}{2}\Big|_0^{\pi/2}$$

2/25

EX. FIND VOL OF A SOLID BENEATH $z = 6 - x - 2y$ AND ABOVE $[0,2] \times [0,1]$

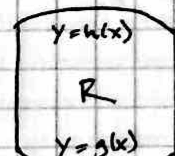


$$\text{VOLUME} = \iint_R (6 - x - 2y)\, dA = \int_0^1 dy \int_0^2 dx\, (6 - x - 2y)$$

2/28 13.2

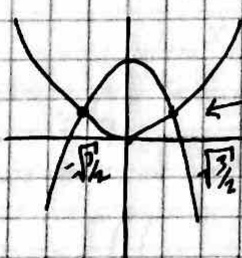
INTEGRATION OVER NON-RECTANGULAR AREAS

$$\iint f(x,y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx$$



$a \le b$
$g(x) \le y \le h(x)$

EX. $\iint_R x y^2\, dA$ R IS BOUNDED BY $y = x^2$, $y = 3 - x^2$



← FIND POINTS OF INTERSECTION

$-\sqrt{\tfrac{3}{2}} \le x \le \sqrt{\tfrac{3}{2}}$

$x^2 \le y \le 3 - x^2$

$\Rightarrow \displaystyle\int_{-\sqrt{3/2}}^{\sqrt{3/2}} dx \int_{x^2}^{3-x^2} x y^2\, dy \ldots$

EX. CHANGE THE ORDER OF INTEGRATION

$\sqrt{x} \le y \le x^2$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} f(x,y)\, dx\, dy$$

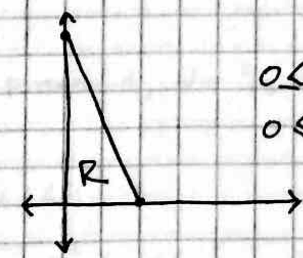
$0 \le y \le 1$

$y^2 \le x \le \sqrt{y}$ MEANS $y \le \sqrt{x}$

$x \le \sqrt{y}$ MEANS $x^2 \le y$

2/28 CONT.

$$\int_0^3\int_0^{6-2x} f(x,y)\,dy\,dx$$



$0 \le y \le 6$
$0 \le x \le 3 - y/2$

3/2

INTEGRATION IN POLAR COORDINATES

$$\iint_R f(x,y)\,dA \qquad (x,y) \rightarrow (r,\theta)$$

$$\iint_{\bar{R}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \Rightarrow \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

↑ JACOBIAN DETERMINATE

3/4 TRIPLE INTEGRATION

CALCULATING MOMENTS

$$\bar{x} = \frac{\iiint_D x\,dV}{x} \qquad \bar{y} = \frac{\iiint y\,dV}{y} \qquad \bar{z} = \frac{\iiint z\,dV}{z}$$

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INTEGRATION USING CYLINDRICAL/SPHERICAL COORDINATES

$$\iiint_D f\,dV = \text{'}dx\,dy\,dz\text{'}$$

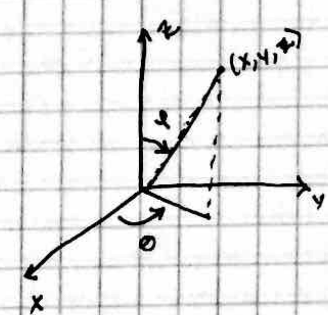
↓ TO CYLINDRICAL COORDINATES

'$r\,dr\,d\theta\,dz$

SPHERICAL COORDINATES

$$\begin{cases} x = \rho \rightarrow \sin\varphi\cos\theta \\ y = \rho \rightarrow \sin\varphi\cos\theta \\ z = \rho \rightarrow \cos\varphi \end{cases} \Rightarrow (\rho, \theta, \varphi)$$

$r = \rho^2 \sin\varphi$



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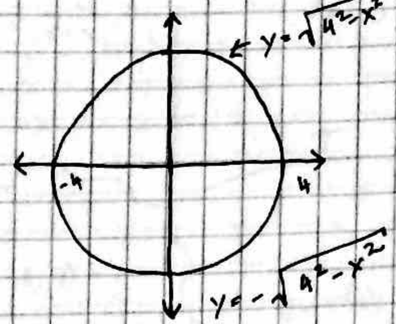
LECTURE CONT.

VOLUME OF A SPHERE OF RADIUS 4, $V = \frac{4}{3}\pi 4^3$



$-4 \le x \le 4$

$-\sqrt{4^2 - x^2} \le y \le \sqrt{4^2 - x^2}$

$z = \sqrt{4^2 - x^2 - y^2}$

VOLUME $= \iiint dV$

SWITCH TO SPHERICAL COORDINATES:

SURFACE OF SPHERE = $\rho = 4$

S: $\theta = 0 \le \theta \le 2\pi$

$\varphi = 0 \le \varphi \le \pi$

$\rho = 0 \le \rho \le 4$

USING TRIPLE INTEGRAL

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

KNOW HOW TO SET UP INTEGRALS!

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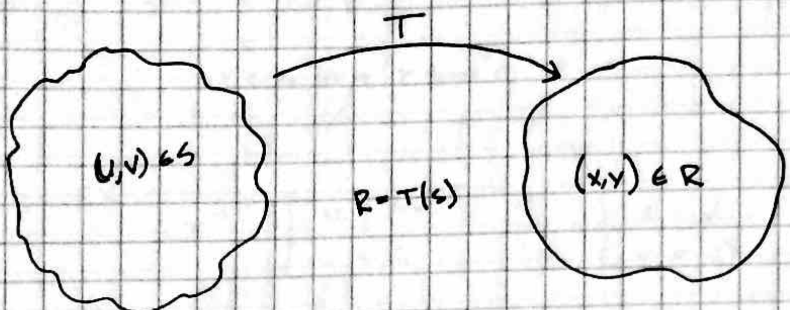
$$\int_{x=a}^{x=b} f(x)\,dx \Rightarrow \begin{cases} g^{-1}(x) = v \\ x = g(u) \\ dx = g'(x)\,du \end{cases} \Rightarrow \int_{g'(a)}^{g'(b)} f(g(x)) \boxed{\frac{\frac{dx}{du}}{g'(x)}}\, du$$

← JACOBIAN



$$\boxed{\iint_R f(x,y)\,dA \Rightarrow \iint_S f(g(u,v)), h(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA}$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$$

JACOBIAN MATRIX / DETERMINANT

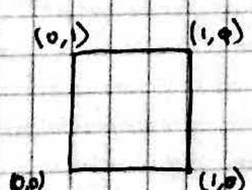
$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\int_0^2 \int_y^{e^y} f(x,y)\,dx\,dy$$

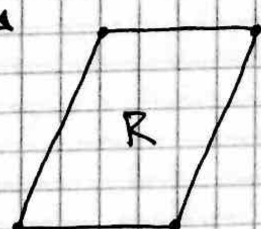
$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

FIND THE JACOBIAN FOR R = T(S) FOR

$$T: \begin{cases} x = 2u + v \\ y = 2u \end{cases} \quad \text{FOR } S = [0,1] \times [0,1]$$



(u,v) $\xrightarrow{T}$	(x,y)
(0,0)	(0,0)
(1,0)	(2,2)
(0,1)	(1,0)
(1,1)	(3,2)



$$\frac{\partial(x,y)}{\partial(u,v)} \Rightarrow \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$2 = \iint_R 1\,dA = \iint_S 1\,|2|\,dA = 2$$

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POLAR COORDINATES:

$$J(r, \theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

EX. $T^{-1}: \begin{cases} u = xy \\ v = x \end{cases}$ $\quad T: \begin{cases} x = v \\ y = \frac{u}{v} \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{u}{v^2} \end{vmatrix}$$

EX. $T: \begin{cases} x = 2uv \\ y = u^2 - v^2 \end{cases}$

$2 \leq x \leq 4$

$\frac{1}{2}x \leq y \leq \sqrt{x}$

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REVIEW:

COORDINATE SYSTEMS

LINES:

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

PLANES $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$

REVIEW PROJECTIONS!

CROSS PRODUCTS GIVE AREA OF PARALLELOGRAM

VELOCITY
REVIEW
$\vec{T}, \vec{N}, K$

MULTIVARIABLE:
REVIEW LEVEL CURVES, SURFACES

PARTIAL DERIVATIVES, FIRST/SECOND ORDER

REVIEW HESSIAN MATRICES

LAGRANGE MULTIPLIERS

$\Delta f = \lambda \nabla g$ NOT INTERESTED IN λ, UNLESS IT CAN HELP FIND x, y

TRANSFORMATIONS

$\mathbb{R}^2 \to \mathbb{R}^2$
JACOBIAN MATRICES