MATH 236

$\frac{dp}{dt} = rp$

$\frac{dp}{dt} = rp - h = 0$

$p = \frac{h}{r}$

$P_0 > 900$ P INCREASES

$P_0 < 900$ P DECREASES

UNSTABLE EQUILIBRIUM SOLUTION

SECTION 1.2 : SOLUTIONS TO SOME ODE's

$\frac{dy}{dt} = ay - b$ HAS SOLUTION $\underbrace{y(t) = Ce^{at} + b/a}_{\text{GENERAL SOLUTION}}$

ODE + INITIAL CONDITION = INITIAL VALUE PROBLEM

IF $C = 0 \Leftrightarrow y_0 = b/a = y^*$

256 RECITATION 9-29-2011

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KIDD 292B

MLC: 9 - 11:55 TUESDAY

QUIZ ALMOST EVERY WEEK, UNLESS TEST OR HOLIDAY
QUIZ TO COVER MATERIAL ~~DAY~~ WEEK BEFORE BUT NOT DAY BEFORE

$$\frac{dy}{dt} = ay + b$$

$$\frac{dy}{y - \frac{b}{a}} = a\,dt \qquad \int \frac{1}{y - \frac{b}{a}}\,dt = \int a\,dt$$

$$= \ln\left(y - \frac{b}{a}\right) = at + C$$

$$y - \frac{b}{a} = Ke^{at}$$

$$\boxed{y = Ke^{at} + \frac{b}{a}}$$

WITH $y(0) = y_0$

$$y_0 = K + \frac{b}{a}$$

$$K = y_0 - \frac{b}{a}$$

$$y = \left(y_0 - \frac{b}{a}\right)e^{at} + \frac{b}{a}$$

a AND b ARE CONSTANTS THAT ARE DEFINED BY THE PROBLEM.
K IS DETERMINED BY INITIAL CONDITION

$$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2$$

$$\frac{dV}{dt} = -rA = -KV^{2/3}$$

$$A = c\left(\frac{4}{3}\pi r^3\right)^{2/3} = cV^{2/3}$$

FIND A WAY TO EXPRESS AREA IN TERMS OF VOLUME.
FIND EQUATION FOR SURFACE OF A SPHERE

MATH 256

SECTION 1.3

CLASSIFICATIONS

ODE VS. PDE:

$$\frac{dy}{dt} + ay = f(t)$$

PDE:

$$\frac{\partial y}{\partial t} + \frac{a\,\partial y}{\partial x} = 0$$

EQUATION VS. SYSTEM:

$$\frac{dy}{dt} + ay = f(t) \quad / \quad \vec{y} + A\vec{y} = \vec{f}$$

MATRIX

ORDER:

HIGHEST ORDER DERIVATIVE THAT APPEARS IN THE EQUATION

LINEAR AND NONLINEAR:

LINEAR DE MUST BE LINEAR IN THE SOLUTION AND ITS DERIVATIVES.

LINEAR:

$y' + ay = 0$

NONLINEAR:

$(y')^2 + y = 0$

$y' \cdot y = 0$

Y TIMES ANY OF ITS DERIVATIVES IS NON-LINEAR

CHAPTER 2:

1ST ORDER ODE

- GENERAL: $y' = f(t,y)$

1ST ORDER LINEAR:

$y' + p(t)y = g(t)$

EX. 1

$p(t) \equiv 0 \qquad y' = g(t)$

$$y(t) = \int^t g(s)\,dt + C$$

EX. 2

$p(t) \equiv a$

$y' + ay = g(t)$

$$\mu(t)y' + a\mu(t)y(t) = \mu(t)g(t)$$

IDEA: MULT. BOTH SIDES BY A FACTOR $\mu(t)$ SO THAT THE ODE LOOKS LIKE (*)

IF WE CAN FIND μ SUCH THAT $\frac{d\mu}{dt} = a\mu$ AND OUR ODE LOOKS LIKE PRODUCT RULE

$\mu(t) = e^{at}$

256, SEPT. 30

Ex. 2

$$\frac{dy}{dt} - 2y = 4-t \qquad \mu = e^{-2t} \text{ WHERE } a = -2$$

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = e^{-2t}(4-t)$$

$$\int \frac{d}{dt}\left[e^{-2t}y\right] = \int 4e^{-2t} - te^{-2t}$$

$$e^{-2t}y = -2e^{-2t} + \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-2t} + C$$

$$y = -2 + \frac{1}{2}t + \frac{1}{4} + Ce^{2t}$$

$$\boxed{y = \frac{-7}{4} + \frac{1}{2}t + Ce^{2t}}$$

CRITICAL $y(0) = y_0$

Ex. 3

$y' + p(t)y = g(t)$ INTEGRATING FACTOR SAYS $\mu y' + p\mu y = \mu g$

IF $\mu' = p(t)\mu(t)$, SUPPOSE $\mu(t) = \exp(x)$

$$\frac{dx}{dt} = p(t)\mu(t)$$

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IF:

$y' + p(t)y = g(t)$

$y = \frac{1}{\mu}\left[\int \mu g + c\right]$ WHERE $\mu = \exp(\int p)$

Ex. $ty' + 2y = 4t^2$

$$y' + \underbrace{\left(\frac{2}{t}\right)}_{p(t) = 2/t} y = 4t$$

$$\mu = \exp\left(\int \frac{2}{s}\,ds\right) = \exp(2\ln|s| + c)$$

$$= \exp(2\ln|s|)\exp(c)$$

$$\exp(\ln t^2)\exp(c)$$

$\mu = ct^2$ ← INTEGRATING FACTOR

PLUG INTO FORMULA:

$$y = \frac{1}{t^2}\left[\int (t^2)(4t)\,ds + c\right]$$

$$= \frac{1}{t^2}\left[t^4 + c\right]$$

$$t^2 + \frac{c}{t^2}$$

DISCONTINUOUS AT $t=0$, EXCEPT WHEN $C=0$

INTERVAL OF DEFINITION $(-\infty,0)$ OR $(0,\infty)$ AT 0. ONE SIDE MUST BE THE SOLUTION TO THE DEFINITION

SECTION 2.2: SEPERABLE EQUATIONS (POSSIBLY NON-LINEAR)

DEFINITION OF SEPERABLE

$$\frac{dy}{dx} = \frac{m(x)}{n(y)} \quad \text{OR} \quad \frac{dy}{dx} n(y) - m(x) = 0 \quad \Rightarrow \quad n(x)\,dx = m(y)\,dy$$

METHOD: SEPARATION OF VARIABLES

IDEA:

LET

$$M(x) = \int m(x)\,dx \quad \text{AND} \quad N(y) = \int n(y)\,dy$$

CONSIDER:

$$\frac{d}{dx}\left[N(y) - M(x)\right]$$

$$= N'(y)\frac{dy}{dx} - M'(x) \quad \Rightarrow \quad n(y)\frac{dy}{dx} - m(x) = 0$$

$$\int n(y)\,dy = \int m(x)\,dx + C$$

EX. $\dfrac{3x^2 + 4x + 2}{2(y-1)}$; $y(0) = 1$

$$\int 2(y-1)\,dy = \int 3x^2 + 4x + 2\,dx + C$$

$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + C}$ IMPLICIT GENERAL SOLUTION

$$(-1)^2 - 2(-1) = 0 + 2(0)^2 + 2(0) + C$$

$$1 + 2 = C$$

$C = 3$ $\boxed{y(0) = -1,\ C = 3}$ PARTICULAR SOLUTION

QUIZ ON 2.1 AND 2.2

MATH 256 SECTION 2.4

- PROPERTIES OF SOLUTIONS TO 1^{ST} ORDER ODE'S
- DIFFERENCES BETWEEN LINEAR AND NON-LINEAR

QUESTIONS:

1. EXISTENCE / UNIQUENESS
2. INTERVAL OF DEFINITION
3. GENERAL SOLUTION
   - CAN WE ALWAYS ~~RIGHT~~ WRITE ALL SOLUTIONS IN TERM OF SOME CONSTANT
4. IMPLICIT SOLUTIONS
   - WHEN CAN YOU NOT SOLVE EXPLICITLY

EXISTANCE UNIQUNESS

IF P AND $G \in C[I]$ $I = (\alpha, \beta)$ AND $t_0 \in I$ THEN $\exists!$ SOLUTION Y TO THE IVP

THERE EXISTS A UNIQUE

$\forall t \in I$

FOR ALL

EX. 1 FIND THE LARGEST INTERVAL IN WHICH A UNIQUE SOLUTION IS CERTAIN TO EXIST.

$$ty' + 2y = 4t^2 \qquad y(1) = 2$$

$$y' + \left(\frac{2}{t}\right)y = 4t$$

$P(t)$ NOT CONTINUOUS AT ZERO

CANIDATE INTERVALS:

$(-\infty, 0)$ OR $\underbrace{(0, \infty)}_{\text{CONTAINS INITIAL } t}$

$$\boxed{= (0, \infty)}$$

PROOF: EXISTANCE PROOF

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s) g(s)\, ds + y_0\right] \; ; \; \mu(t) = \exp\left(\int_{t_0}^{t} P(t)\, ds\right)$$

UNIQUENESS: ASSUME Y_1 AND Y_2 ARE BOTH SOLUTIONS

$$y_1' + P(t)y_1 = g(t) = y_2' + P(t)y^2$$

$$(y_1 - y_2)' + P(t)(y_1 - y_2) = 0$$

$$\boxed{\text{SOLUTIONS TO DIFFERENTIAL EQUATIONS CANNOT CROSS}}$$

MATH 256

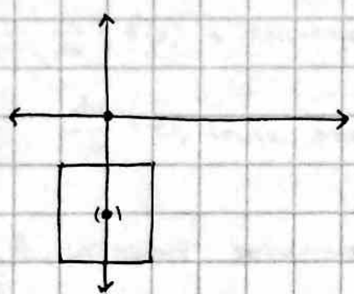
NON-LINEAR 1ST ORDER $y' = f(t,y)$

THRM 2.4.2

IF f & $\frac{\partial f}{\partial y} \in C[R]$, $R = [\delta, \gamma]$, THEN $\exists!$ SOLUTION y TO IVP

$y' = f(t,y)$ $\quad y(t_0) = y_0 \quad \forall\, t \in (t_0 - h, t_0 + h) \subset (\alpha, \beta)$

EX. 2 $\frac{dy}{dx} = \frac{5x^2 + 4x + 2}{2(y-1)}$ $\quad y(0) = -1$



$y = 1 \pm (x^3 + 2x^2 + 2x)^{1/2}$

MATH 256: 10-12-2011

Ex. 3 $y' = y^{1/3}$, $y(0) = 0$

$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$, DISCONTINUOUS AT $y = 0$

SECTION 2.5 AUTONOMOUS ODE'S

$\frac{dy}{dt} = f(t,y) \Rightarrow \frac{dy}{dt} = -p(t)y + g(t) \overset{\text{or}}{\Rightarrow} \frac{dy}{dt} = \frac{-m(t)}{n(y)}$

NOW WE LOOK AT:

$\frac{dy}{dt} = f(y) \rightarrow$ AUTONOMOUS. CAN SOMETIMES SOLVE FOR EQUILLIBRIA ANYLITICALLY.

$\frac{dy}{dt} = 0$, SOLVE FOR y

4. DIFFERENT POPULATION MODELS

1. EXPONENTIAL GROWTH
2. LOGISTIC GROWTH
   - DENSITY DEPENDENT GROWTH RATE.
3. CRITICAL THRESHOLD
   - IF TOO FEW, POP. 'CRASHES'
4. = 2 + 3

---

1. $y' = ry$, $r > 0$ GOOD MODEL FOR SMALL POPUTATIONS

2. $y' = R(y)y$
   a. IF $y \approx 0$, $R(y) \approx r$
   b. $R'(y) < 0$
   c. LINEAR

STABILITY: ASYPTOTICALLY STABLE IF y_0 NEAR y^* IMPLIES $y(t) \rightarrow y^*$

MAXIMUM SUSTAINABLE YIELD:

$$h^* = f\left(\frac{K}{2}\right) = \frac{rK}{4}$$

SOLVING NOT NECESSARY ~~FOR~~ S

SECTION 2.6 EXACT EQUATIONS

RECALL

$$n(y)\frac{dy}{dx} - m(x) = 0 \quad \Leftrightarrow \quad \frac{d}{dx}\left[N(y) - M(x)\right] = N'(y)\frac{dy}{dx} - m(x)$$

CONSIDER,

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0$$

$$\frac{d}{dx}\left[\Psi(x,y)\right] = \frac{\partial y}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}$$

THRM 2.6.1

SUPPOSE $N(x,y)\frac{dy}{dx} + M(x,y) = 0$

THE ODE IS EXACT IFF:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

EXACT

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial x} &= M \\ \frac{\partial \Psi}{\partial y} &= N \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} \Psi_{xy} &= M_y \\ \Psi_{yx} &= N_x \end{aligned}\right\} \Rightarrow M_y = N_x$$

METHOD FOR SOLVING EXACT ODE

0. GIVEN $N(x,y)\frac{dy}{dx} + M(x,y) = 0$

TEST $M_y = N_x$

1. $\Psi(x,y) = \int M(x,y)\,dx + h(y)$

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + h'(y) = N.$$

3. $$h = \int h' \Rightarrow \psi$$

Ex. $$\underbrace{\left(y\cos(x) + 2xe^y\right)}_{M} + \underbrace{\left(\sin(x) + x^2e^y - 1\right)}_{N} y' = 0$$

0. $M_y = \cos(x) + 2xe^y \qquad N_x = \cos(x) - 2xe^y$

1. $$\psi = \int m\,dx + h(x)$$
$$= y\sin(x) + \cancel{2}x^2e^y - h(y)$$

2. $$\frac{\partial \psi}{\partial y} = \sin(x) + x^2e^y + h'(y)$$
$$= \sin(x) + x^2e^y - 1$$
$$\Rightarrow h'(y) = -1$$

3. $$\boxed{\psi(x,y) = y\sin(x) + x^2e^y - y + c = C}$$

MATH 256

METHOD FOR EXACT ODE.

$M + Ny' = 0$; EXACT $\Leftrightarrow M_y = N_x$ $\quad C = \psi$; $\frac{\partial}{\partial y}\left(\int m\,dx + h\right) = N$

EXACT:

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

INTEGRATING FACTOR TO SOLVE

$$M + Ny' = 0 \rightarrow \mu M + \mu N y' = 0 \quad \underbrace{\mu_y M + \mu M_y = \mu_x N + \mu N_x}_{\text{PDE}}$$

TRY A $\mu = \mu(x)$, THEN A $\mu = \mu(y)$

$$\mu M_y = \mu_x N + \mu N_x \Leftrightarrow \mu' = \frac{M_y - N_x}{N}\mu \quad \text{OR} \quad \mu' = \frac{N_x - M_y}{N}\mu$$

$$\frac{(2x + y) - (3x + 2y)}{3xy + y^2} = \frac{-x - y}{y(3x + y)} \leftarrow \text{DOESN'T WORK}$$

$$\mu' = \frac{M_y - N_x}{N}\mu = \frac{x + \cancel{y}}{x(\cancel{x+y})}\mu = \frac{\mu}{x}; \quad \mu' = \frac{\mu}{x} \quad \text{SOLVING, } \mu = x$$

CHAPTER 3: SECOND ORDER LINEAR

GENERAL SECOND ORDER LINEAR

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

SECTION 3.1 CONSTANT COEFF. HOMOGENEOUS

$- G(t) = 0$

IVP = ODE + IC

2^{ND} ORDER $\rightarrow$ 2 INITIAL CONDITIONS

MATH 256 SECTION 3.3

GSOLCCHODE

$ay'' + by' + cy = 0 \Rightarrow ar^2 + br + c = 0$ WHERE $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

REAL DISTINCT ROOTS $y_1 = e^{r_1 t}, \; y_2 = e^{r_2 t}$

COMPLEX ~~PART~~ ROOTS $b^2 < 4ac$ $\qquad \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$r_{1,2} = \lambda \pm i\omega$

$y_{1,2} = e^{(\lambda \pm i\omega)t}$

$e^{i\omega t}$

$f(t) = f(0) + f'(0)t + \dfrac{f''(0)}{2!}t^2 + \dfrac{f'''(0)}{3!}t^3$

$e^{i\omega t} = 1 + t(i\omega) + \dfrac{t^2(i\omega)^2}{2!} + \dfrac{t^3(i\omega)^3}{3!} + \dfrac{t^4(i\omega)^4}{4!}$

$= 1 + i\omega t - \dfrac{(\omega t)^2}{2!} - i\dfrac{(\omega t)^3}{3!} + \dfrac{(\omega t)^4}{4!}$

$e^{i\theta} = \cos\theta + i\sin\theta$

SEPARATE ODD/EVEN TERMS:

$$\underbrace{\left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!}\right)}_{\cos(\omega t)} + i\underbrace{\left(\omega t - \frac{(\omega t)^3}{3!}\right)}_{\sin(\omega t)} = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

SOLVE FOR:

$y_1 = e^{\lambda t}\left[\cos(\omega t) + i\sin(\omega t)\right]; \quad y_2 = e^{\lambda t}\left[\cos(\omega t) - i\sin(\omega t)\right]$

WHERE $\lambda = b^2 - 4ac$

MATH 256

EX. 1 $y'' + y' + y = 0$

$r^2 + r + 1$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \underbrace{\left(\frac{-1}{2}\right)}_{\lambda} \pm \underbrace{\left(\frac{\sqrt{3}}{2} i\right)}_{\omega}$$

$y_1 = e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)$

$y_2 = e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right)$

GENERAL SOLUTION FORMAT:

$u(t) = e^{\lambda t} \cos(\omega t)$

$v(t) = \dfrac{e^{\lambda t} \sin(\omega t)}{\omega}$

EX.

$y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \quad r = \pm 3i$

$y_1 = \cos(3t)$

$y_2 = \sin(3t)$

METHOD OF REDUCTIONAL ORDER

$L[y] = y'' + p(t)y' + q(t)y = 0$

$y_1 w' + (2y' + p(t)y_2)w = 0$; $v' = w$

FIND w

$v = \int w$ $\quad y = vy_1$

EX. $2t^2 y'' + 3ty - y = 0$; GIVEN $y_1 = \frac{1}{t}$ FIND A y_2

SOL: LET $y = v(t)y_1$, NOTE $y_1' = -t^{-2}$

$$\frac{1}{t}w' + \left[\frac{4}{2}\left(\frac{-1}{t^2}\right) + \left(\frac{3}{2t}\right)\left(\frac{1}{t}\right)\right]w = 0$$

$$w' = \frac{1}{2t}w \Rightarrow \frac{dw}{w} = \frac{dt}{2t} \Rightarrow \int \frac{1}{w}dw = \int \frac{1}{2t}dt \Rightarrow \ln|w| = \frac{1}{2}\ln(t) + c$$

$$w = ct^{1/2}$$

NON-HOMOGENEOUS:

$L[y] = y'' + p(t)y' + q(t) = g(t)$

3.5.1

IF Y_1 AND Y_2 TO $[L] = g(t)$, THEN $Y_1 - Y_2$ SOLVES $L[y] = 0$

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SECTION 3.5 NON-HOMOGENEOUS

3.5.2 GENERAL SOLUTION TO NON-HOMOGENEOUS

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{HOMO. SOLUTION}} + Y$$

ONE SOLUTION TO $L[y] = g(t)$

STEPS:

1.) FIND ~~NON~~ HOMO. SOLUTION

2.) FIND Y

3.) 1.) + 2.)

EX. FIND Y FOR

$$y'' - 2y' - 3y = 3e^{2t}$$

ASSUME $Y = Ae^{2t}$ $\Rightarrow$ $Y = -e^{2t}$

$$y' = 2Ae^{2t}$$
$$y'' = 4Ae^{2t}$$

$$4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t}$$

$A = -1$

EX. 2

$$y'' + 2y' + 5y = 3\sin(2t)$$

ASSUME $Y = A\sin(2t) + B\cos(2t)$

$$Y' = 2A\cos(2t) - 2B\cos(2t)$$
$$y'' = -4A\sin(2t - 4B\cos(2t)$$

$$(-4A\sin(2t) - 4B\cos(2t) + 2(2A\cos(2t) - 2B\sin(2t)) + 5(A\sin(2t) + B\cos(2t)) = 3\sin(2t)$$

$A - 4B = 3$ $\quad$ $B = -4A \Rightarrow -12/17$

$4A + B = 0$ $\quad$ $A = 3/17$

$$Y = \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)$$

MATH 256

$y'' + y' - 2y = 2t$

GUESS $Y = At + B$

$y' = A$

$y'' = 0$

$\Rightarrow 0 + A - 2(At+B) = 2t$

$A = -1$

$B = -1/2$

$y = -t - 1/2$

SO FAR WE HAVE

$G(t)$	Y
$Ce^{\alpha t}$	$Ae^{\alpha t}$
$C\cos(\omega t) + D\sin(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
$P_n(t)$	$P_n(t)$

NOTE: IF $G(t) = g_1(t) g_2(t)$ THEN $Y = Y_1 Y_2$

* SUPPLIMENTERY PROBLEM *

EX. $y'' - 2y' - 3y = 3te^{2t}$ SO $Y = (At+B)(Ce^{2t})$

IF $g(t) = g_1(t) + g_2(t)$

$Y = Y_1 + Y_2$

WATCH OUT FOR GUESS SOLUTIONS THAT ARE SOLUTION TO THE HOMOGENIOUS ODE.

INSTEAD: $Y = At^2 e^{-t} \Rightarrow y'' + 2y' + y = 2e^{-t}$ WHERE $\underbrace{y_1 = e^{-t} \text{ AND } y_2 = te^{-t}}_{\text{SOLUTIONS TO HOMOGENEOUS ODE}}$

SECTION 4.1

GENERAL FORM FOR NTH ORDER LINEAR

$$L_n[y] = \frac{d^n y}{dt^n}$$