MATH 482 1/9/2012

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'OR' MATH → FACULTY → SHOWALTER → 582

PARTIAL DIFFERENTIAL EQUATIONS

- HEAT EQUATION
- WAVE EQUATION
- LAPLACE EQUATION

CHAPTERS 10/11 + POWER SERIES CHAPTER

2 TESTS 50 MIN EACH

SCORE ON MIDTERMS + FINAL SCORE × 2 : TAKES THE BEST OF 3
- MOST PROBLEMS WILL BE SIMILAR TO THE BOOK (NOT NECC. FROM THOUGH)
+ HW POSTED ONLINE

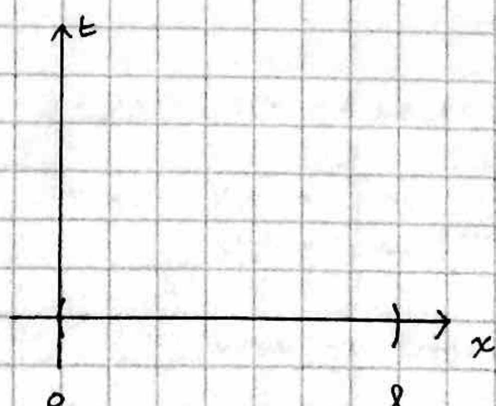
WILL POST OFFICE HOURS WHEN HE FINDS OUT SCHEDULE

NOTES:

HEAT EQUATION / DIFFUSION EQUATION

$$\frac{\partial u}{dt} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \; ; \quad u(x,t)$$

EQUATION DESCRIBES TEMP. AS A FUNCTION OF POSITION $x \in (0, \ell)$ IN BAR



INITIAL CONDITIONS:

$u(x,0) = f(x)$ ← KNOWN

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(\ell,t) = 0 \end{array}\right\} \text{BOUNDERY CONDITIONS}$$

PDE + IC + BC = INITIAL - BOUNDERY VALUE PROBLEM

1/9/2012

SOLVE : $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2}$ SOLUTION GUESS $u = X(x)T(t)$

PLUG INTO PDE :

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \Rightarrow \frac{T'(t)}{\alpha^2 T^2(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\Rightarrow X''(x) + \lambda X(x) = 0$$

AND

$$T'(t) + \alpha^2 \lambda T(t) = 0$$

FINDING SOLUTIONS TO PDE'S IS EASY, FINDING SOLUTIONS THAT SATISFY IC/BC IS TOUGH

PLUG IN IC

$$X''(x) + \lambda X(x) = 0$$
$$X(0) = 0 \quad X(\ell) = 0$$
$\Rightarrow$ TWO-POINT BOUNDERY VALUE PROBLEM

$\underline{\lambda < 0:}$ SET $\lambda = -\mu^2$ ODE BECOMES $X'' - \mu^2 X = 0$

SO $X = c_1 e^{\mu x} + c_2 e^{-\mu x} \rightarrow X(0) = c_1 + c_2 = 0$

$X(\ell) = c_1 e^{\ell \lambda} + c_2 e^{-\ell\lambda}$

$c_1 + c_2 = 0$ SO $\underbrace{x = 0 \ldots}_{}$

INCONCLUSIVE

$\underline{\lambda = 0}$

$$X'' = 0$$
$$X = c_1 + c_2 x$$
$$X(0) = \underbrace{c_1 x = 0}$$

$X = 0 \ldots$ INCONCLUSIVE

$\underline{\lambda > 0}$ $X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

BC'S: $X(0) = c_1 = 0$

$X(\ell) = c_2 \sin(\sqrt{\lambda}\, \ell) = 0$

NEED TO SATISFY $\sqrt{\lambda}\, \ell = n\pi$ $n = 1, 2, 3 \ldots$ $\lambda = \left(\frac{n\pi}{\ell}\right)^2$

$$X_n(x) = \sin\left(\frac{n\pi}{\ell} x\right)$$

1/9/2012

$$T'(t) + \alpha^2 \lambda T(t) = 0$$

$$T_n(t) = e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}$$

$$\boxed{u_n(x,t) = e^{-\left(\frac{n\pi\alpha}{\ell}\right)t} \sin\left(\frac{n\pi}{\ell}x\right)}$$

SO

$$\boxed{u(x,t) = \sum_{n=1}^{N} c_n e^{-\left(\frac{n\pi\alpha}{\ell}\right)t} \sin\left(\frac{n\pi}{\ell}x\right)}$$

← SOLUTION TO PDE AND BC.

INITIAL CONDITIONS:

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{\ell}x\right) = f(x)$$ ← FORIER SERIES

SECTION 10.5 (LOOK @ 7,8)

JANUARY 11, 2012

10.5: 7, 8, 9
10.1: 1, 3, 5, 15

$$\boxed{\begin{array}{l} X'' + \lambda X = 0 \\ X(0) = 0,\ X(L) = 0 \end{array}}$$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \ldots$

$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \qquad T_n(t) = e^{-\left(\frac{n\pi}{\ell}t\right)^2}$

$$u(x,t) = \sum_{n=1}^{N} C_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}\right)t \xrightarrow{\text{SET } t=0} \sum_{n=1}^{N} C_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$X_n(x) = X_m(x) \Rightarrow \int_0^\ell X_n(x)\, X_m(x)\,dx \Rightarrow \lambda_n \int_0^\ell X_n(x)\, X_m(x)\,dx$$

$$= -\int_0^\ell X_n''(x)\, X_m(x)\,dx \overset{\text{IBP}}{=}$$

$$\frac{d}{dx}(X_n' X) = X_n'' X_m \cancel{dx} + X_n' X_m' \cancel{dx} \Rightarrow (X_n' X_n) = \int X_n'' X_m \cancel{dx} \int X_n X_m\,dx$$

$$\int_0^\ell X_n'(x) X_m'(x)\,dx = -\int_0^\ell X_n X_m''\,dx = \lambda_m \int_0^\ell X(x)_n X(x)_m\,dx$$

$$\boxed{\begin{array}{c} \int_0^\ell X_n(x)\, X_n(x)\,dx = 0 \\ \text{IF } m \neq n \end{array}} \Rightarrow \text{IE. } \{X_n(x)\} \text{ ARE } \underline{\text{ORTHAGONAL}}$$

(REPLACING VECTORS W/ FUNCTIONS

FIND C_n:

$$f(x) = \sum_{n=1}^{\infty} C_n X_n(x) \Rightarrow \int X_m(x) f(x)\,dx = \sum_{n=1}^{\infty} C_n \int_0^{l} X_m(x) X_n(x)\,dx$$

THE ONLY LOCATION WHERE THERE IS A NON-ZERO SCALAR PRODUCT IS WHERE $\underline{\underline{m=n}}$ (WE JUST SHOWED THAT)

SO,

$$= C_n \int_0^l X_m^2\,dx$$

TRIG IDENT. TO INTEGRATE:

$$\cos^2 A + \sin^2 A = 1$$
$$\cos^2 A - \sin^2 A = \cos(2A)$$

$$\sin^2 A = \frac{1}{2}\left(1 - \cos(2A)\right)$$

$$\underbrace{\int_0^l \sin^2\left(\frac{n\pi x}{L}\right) dx}_{\text{INTEGRATES AWAY}} = \frac{l}{2}$$

$$C_n \int_0^l X_m^2\,dx = C_m \frac{l}{2} \Rightarrow \boxed{C_m = \frac{2}{l} \int f(x) \sin\left(\frac{n\pi x}{l}\right) dx}$$

$f(x) = 1$ so;

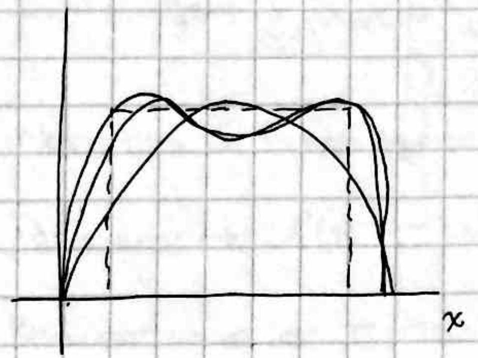
$$C_n = \frac{2}{L} \int_0^l \sin\left(\frac{n\pi x}{L}\right) dx \Rightarrow -\left(\frac{2}{l}\right)\left(\frac{l}{n\pi}\right)\cancel{\cos} = \left(\frac{2}{l}\right)\left(\frac{l}{n\pi}\right)\left(1 - \cos(n\pi)\right)$$

EVAL. FROM $0 \to L$

$$\left(1 - \cos(n\pi)\right)$$

$$\begin{cases} 2 & n\text{, odd} \\ 0 & n\text{, EVEN} \end{cases} \Rightarrow C_n = \frac{4}{n\pi} \text{ IF } n = 1, 3, \ldots$$

$$1 = \frac{4}{\pi}\sum \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right), \quad \cancel{0 < x < L} \quad 0 < x < l$$



WANTS TO BE =1, MUST VANISH @ ENDPOINTS HOWEVER

$$\cos^2\theta + \sin^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos^2\theta + 1 - \cos^2\theta = \cos(2\theta)$$

$$\cos^2\theta - \cos^2\theta = \cos(2\theta) - 1$$

$$-\cos^2 +$$

READING NOTES, 1-18-2012

FORIER SERIES → ANALOGOUS TO TAYLOR SERIES, PROVIDE A MEANS OF EXPRESSING COMPLICATED FUNCTIONS AS SUMS OF SINES AND COSINES.

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$$

PROPERTIES OF FORIER SERIES:

PERIODICITY → $f(x+T) = f(x)$

SMALLEST VALUE OF T FOR WHICH THE ABOVE HOLDS IS CALLED THE FUNDEMENTAL FREQUENCY. ANY INTERER MULTIPLE OF T IS ALSO A PERIOD.

ORTHAGONALITY:

THE STANDARD INNER PRODUCT (u,v) OF TWO REAL-VALUED FUNCTIONS ON $\alpha \leq x \leq \beta$ IS DEFINED BY:

$$(u,v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx$$ → IF THIS INTEGRAL IS 0, THE FUNCTIONS ARE

SAID TO BE ORTHAGONAL. IF EACH DISTINCT PAIR OF FUNCTIONS IS ORTHAGONAL, THE FUNCTIONS ARE SAID TO BE MUTUALLY ORTHAGONAL

EULER-FORIER FORMULAS:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{AND} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

JANUARY 23, 2012

10.2 13, 14, 16
10.3 1, 2, 5

$$f(x) = \frac{a_0}{2} + \sum_{n=1} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n}{L}x\right)dx \quad \text{AND} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

EX.

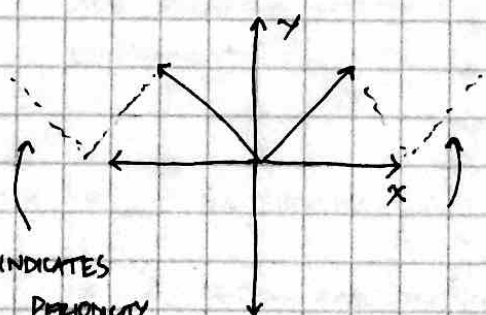
$$f(x) = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{L}x\right) + \frac{1}{3}\sin\left(\frac{\pi}{L}x\right) \ldots\right)$$

2L PERIODIC

ODD $f(x)$ = ODD FUNCTION FORIER SERIES (SIN TERMS)

EX. 2

$f(x) = |x|$



INDICATES PERIODICITY

2L - PERIODIC

EVEN ODD

$$b_n = \frac{1}{L}\int_{-L}^{L} |x| \sin\left(\frac{n\pi}{L}x\right)dx = 0$$

MOST INTEGRATE TO ZERO

$$a_0 = \frac{1}{L}\int_{-L}^{L} |x|(1)\,dx = \frac{2}{\ell}\int_0^L x\,dx = \ell$$

EVEN TRIG, EVEN $f(x)$

$$a_n = \frac{1}{L}\int_{-L}^{L} |x| \cos\left(\frac{n\pi}{L}x\right)dx$$

SYMETRY

$$= \frac{2}{\ell}\int_{0}^{L} x\cos\left(\frac{n\pi}{L}x\right)dx$$

$$\frac{2}{L}\left[x\frac{l}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\right]_0^L - \frac{2}{L}\int \frac{L}{n\pi}\sin\left(\frac{n\pi}{l}x\right)dx$$

LIMITS CAUSE TO $=0$

$$= -\frac{2}{L}\int \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{n\pi}\left[\frac{-l}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{x=0}^{x=L} = \underbrace{\frac{-2l}{(n\pi)^2}\left(1-\cos(n\pi)\right)}$$

$= 2,\ n = 1,3,5$

$= 0,\ = 0,2,4$

$$= \begin{cases} \frac{-4L}{(n\pi)^2} & ;\ n \text{ odd} \\ 0 & ;\ n \text{ EVEN} \end{cases}$$

$$f(x)_2 = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\left(\frac{\pi}{L}x\right) + \frac{1}{3^2}\cos\left(\frac{3\pi}{L}x\right) + \frac{1}{5^2}\cos\left(\frac{5\pi}{L}x\right) \ldots\right)$$

JANUARY 25, 2012

$$f(x) = \begin{cases} 1 & \text{ON } (0, \ell) \\ -1 & (-\ell, 0) \end{cases} \qquad b_n \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ EVEN} \end{cases}$$

$$\frac{4}{\pi} \sin\left(\frac{\pi}{\ell} x\right) + \frac{1}{3} \sin\left(\frac{3\pi}{\ell}\right) \ldots$$

FOR $|x|$ ON $(-\ell, \ell)$

$$= \frac{\ell}{2} - \frac{4\ell}{\pi^2}\left(\cos\left(\frac{\pi}{\ell}x\right) \; \frac{1}{2^2}\cos\left(\frac{\pi}{\ell}x\right) \ldots\right) \Rightarrow$$ DERIVATIVE OF ABOVE

THEORUM: 1 PIECEWISE

IF f IS 2ℓ PERIODIC, AND IF f AND f' ARE PIECEWISE CONTINUOUS, THEN AT EACH $x \in \mathbb{R}$, THEN THE FORIER SERIES OF $f(x)$ CONVERGES TO $\frac{f(x^+) + f(x^-)}{2}$. IF FUNCTION IS CONTINOUS ON INTERVAL BEING INSPECTED, THEN $f(x+) = f(x^-)$ SO THE DIFFERENCE $= f(x)$

THEORUM: 2 UNIFORM

IF f IS 2ℓ PERIODIC AND CONTINUOS, BOTH f' AND f'' ARE PIECEWISE CONTINUOUS, THEN THE FORIER SERIES CONVERGES UNIFORMLY.

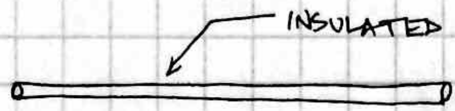
THEORUM: 0 (MEAN SQUARE)

IF f IS SQUARE-INTEGRABLE, IE: $\int_{-\ell}^{\ell} |f(x)|^2 dx < \infty$,

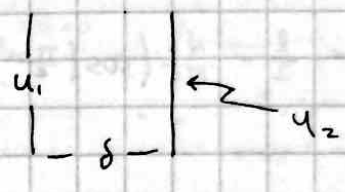
THEN FORIER SERIES IS DEFINED AND $\int_{-L}^{L} |f(x) - \text{SERIES}|^2 dx \rightarrow 0$

JANUARY 27 2012 CONT.

10.6 : 1, 2, 9a, 11a :



FORIERS LAW:



HEAT ENERGY IN $[x, x+h]$

$$\int_x^{x+h} c(s)\rho(s)u(x,t)\,S\,ds$$

SPECIFIC HEAT

$$K\frac{u_1-u_2}{\delta} \Rightarrow -K(x)\frac{\partial u(x,t)}{\partial x}$$

$$= g(x,t)$$

$$\frac{d}{dt}\int_x^{x+h} c(s)\rho(s)u(s,t)\,S\,ds$$

$$= S\,g(x,t) - S\,g(x+h,t) + \int_x^{x+h} S\,f(s,t)\,ds$$

$$= \ldots\quad c(x)\rho(x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(K(x)\frac{\partial u(x,t)}{\partial x}\right)$$

$$= f(x,t) \quad 0<x<\ell,\ t>0$$

$$= \frac{\partial u}{\partial t} - \alpha^2\frac{\partial^2 u}{\partial x^2} \qquad \alpha^2 = \frac{K}{c\rho^2}f(x,t)$$

INITIAL $u(x,0) = u_0(x)$

BOUNDERY CONDITIONS:

$x = \ell \quad u(\ell, t) = \bar{u}_0(t)$

DERIVATION ONLINE

JANUARY 30TH, 2012

TEST #1, FRIDAY, FEB 10TH

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \quad \text{NON-HOMOGENEOUS}$$

$$BC: \begin{cases} u(0,t) = d_0(t) \\ u(\ell,t) = d_\ell(t) \end{cases}$$

IC: $u(x,0) = f(x)$

SOLUTION: $F = 0, \; d_0 = d_\ell = 0$

TRY $u(x,t) = X(x)T(t) \Rightarrow$

$$\boxed{\begin{aligned} &X''(x) + \lambda X(x) = 0 \\ &T' + \lambda T \alpha^2 = 0 \\ &X(0) = X(\ell) = 0 \end{aligned}} \Rightarrow$$

$X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$

$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad \Rightarrow \lambda$ IS EIGENVALUE TO $\Rightarrow T_n(t) = e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}$

SOLUTION W/ BC $= X_n(x)T_n(t) \Rightarrow u(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) T_n(t)$

IC: $\sum_{n=1}^{\infty} c_n X_n(x) = f(x)$

$$c_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

= PROBLEM SOLVED FOR IC/BC

$$\boxed{u = \sum_{n=1}^{\infty} \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \cdot \sin\left(\frac{n\pi x}{\ell}\right)\cdot e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}}$$

COMPLETE SOLUTION

NOW, NON-HOMOGENEOUS BC: (WILL ADD ALL SOLUTIONS LATER)

$$\frac{\partial v}{\partial t} - \alpha^2\frac{\partial^2 v}{\partial x^2} = 0 \qquad \text{BC: } \begin{cases} v(0,t) = d_0(t) \\ v(\ell,t) = d_\ell(t) \end{cases}$$

IC: $v(x,0) = 0$

TRY $w = v - \left[\left[\frac{\ell - x}{\ell}\right]d_0 t + \frac{x}{\ell}d_\ell(t)\right]$, w SATISFIES:

$w(0,t) = 0$, $w(\ell,t) = 0$

IC: $w(x,0) = -\left[\frac{\ell-x}{\ell}d_0 t + \frac{x}{\ell}d_\ell(t)\right]$

W/ SOLUTION: $\frac{\partial w}{\partial t} - \alpha^2\frac{\partial^2 w}{\partial t^2} = \frac{\ell - x}{\ell}d_0'(t) + \frac{x}{\ell}d_\ell'(t)$

$$* \quad w = u - \left[\frac{\ell - x}{\ell}d_0(t) + \frac{x}{\ell}d_\ell(t)\right]$$

$$w(x,0) = f(x) - \left[\frac{\ell-x}{\ell}d_0(t) + \frac{x}{\ell}d_\ell(0)\right]$$

IC:

$$w(x,0) = f(x) - \left[\frac{l-x}{l} d_0(0) + \frac{x}{l} d_l(0) \right]$$

$$v = \sum_{n=1}^{\infty} v_n(t) X_n(x)$$

$$F^*(x,t) = \sum_{n=1}^{\infty} F_n(t) X_n(x) \leftarrow \text{KNOWN}$$

1.) SUBTRACT OF BOUNDARY COND.

2.) SOLVE FOR 'v'

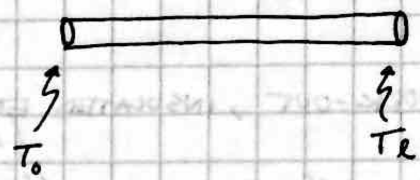
FEBRUARY 1, 2012

<u>10.6</u> 12 a,b; 13 a,c

$$u_t - \alpha^2 u_{xx} = F(x)$$

$$u(0,t) = T_0 \qquad u(\ell,t) = T_\ell$$

$$u(x,0) = f(x)$$



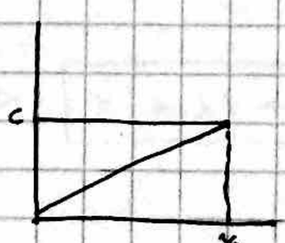
$u(x) = \lim_{t\to\infty} u(x,t)$ MUST SATISFY $\left[\begin{array}{l} -\alpha^2 u'' = F(x) \\ u(0) = T_0 \quad u(\ell) = T_\ell \end{array}\right]$

TWO-POINT BOUNDERY VALUE PROBLEM, TRIVIAL. SOLUTION SOLVES PDE THAT IS INDEPENDENT OF TIME

SOLVE BVP:

$$-v'' = \frac{1}{\alpha^2} F(x) \Rightarrow -v'(x) = \frac{1}{\alpha^2}\int F(s)\,ds \Rightarrow$$

$$-v(x) + c_1 + c_2 x = \frac{1}{\alpha^2}\int_0^x \int_0^c F(s)\,ds\,dc$$



$$\int_0^x \int_s^x \frac{1}{\alpha^2} F(s)\,dc\,ds$$

FINAL:

$$v(x) = c_1 + c_2 x - \int_0^x \frac{x-s}{\alpha^2} F(s)\,ds$$

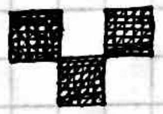
$x = 0, \quad T_0 = c_1$

$x = \ell, \quad T_\ell =$

$$\begin{cases} u_t - \alpha^2 u_{xx} = F(x,t) \\ u(0,t) = T_0(t) \quad u(\ell,t) = T_\ell(t) \\ u(x,0) = f(x) \end{cases}$$

WORK-OUT, INSULATED ENDS PROBLEM

$X'(0,t) = 0 \qquad X'(\ell,t)$



ADD B.

10.7: 1a, 5a, 9

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial t^2} = 0$$ (SOLUTION W/ DIFFERENT BOUNDARY CONDITIONS)

$\Downarrow$

$$\boxed{\begin{array}{l} X'' + \lambda X = 0 \\ X(0) = 0,\ X(\ell) = 0 \end{array}}$$

$X_n = \sin\left(\frac{n\pi}{\ell} x\right)$

WHERE $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$

$$u(x,t) = \sum C_n \sin\left(\frac{n\pi}{\ell} x\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 t}$$

@ $t=0$ (SOLVE FOR IC)

$$u(x,0) = \sum C_n \sin\left(\frac{n\pi}{\ell} x\right) = f(x)$$

FOUND USING FORIER SERIES

CHANGE BOUNDARY CONDITIONS:

$u_x(0,t) = 0,\quad u_x(\ell,t) = 0 \quad u(x,0) = f(x)$

$\Rightarrow$ $\boxed{X'' + \lambda X = 0 \text{ , BUT } X'(0) = 0,\ X'(\ell) = 0}$ NEW EIGENVALUE PROBLEM

$X_n(x) = \cos\left(\frac{n\pi}{\ell} x\right)$, $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $n > 0$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{\ell} x\right) e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}}$$ $\leftarrow$ $\underline{n=0}$

$$C_n = \frac{2}{\ell}\int_0^L f(x)\cos\left(\frac{n\pi}{\ell}x\right)dx = a_n \Leftarrow n=1$$

$$C_n = \frac{2}{\ell}\int f(x)\,dx = \left(\frac{a_0}{2}\right) \Leftarrow n=0$$

EX:

$$\frac{\partial u}{\partial t} - \alpha^2\frac{\partial^2 u}{\partial t^2} = 0 \qquad u(0,t)=0,\ u_x(\ell,t)=0$$

$$u(x,0) = f(x)$$

$$u = X(x)T(t)$$

$$\boxed{X'' + \lambda X = 0 \Rightarrow X(0)=0,\ X'(\ell)=0}$$ E.V. PROBLEM

$$T' + \lambda\alpha^2 E = 0$$

$\lambda = 0 \Rightarrow$ NO NON-ZERO SOLUTION

$\lambda < 0;\quad X = A\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$

@ $X(0) = a = 0$ (FORCES $a=0$)

@ $X'(\ell) = \underbrace{b\sqrt{\lambda}\cos(\sqrt{\lambda}x) = 0}$

HOLDS FOR $\sqrt{\lambda}x = n\pi + \frac{\pi}{2}$;
WHERE $\cos = 0$

$n > 0$;

$$\lambda_n = \left[\left(n+\frac{1}{2}\right)\frac{\pi}{\ell}\right]^2 \Rightarrow X_n(x) = \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\ell}x\right)$$

$$= \sin\frac{(2n+1)\pi x}{2\ell}$$

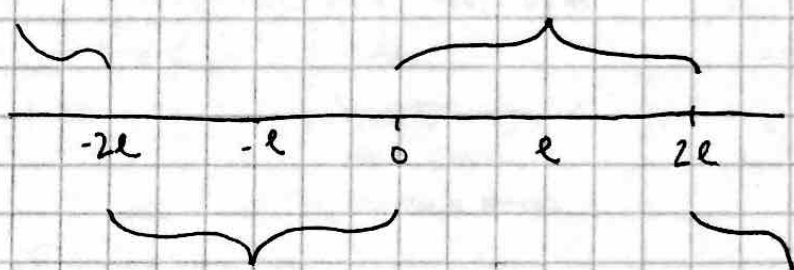
CHECK THAT THEY
SATISFY THE E.V.

$$u(x,t) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right) e^{-\left(\frac{(2n+1)\pi\alpha}{2\ell}\right)^2 t}$$

INITIAL CONDITION:

$$f(x) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{(2n+1)\pi}{2\ell} x\right)$$

GRAPHICALLY, WHAT DOES ORTHAGONALITY MEAN?

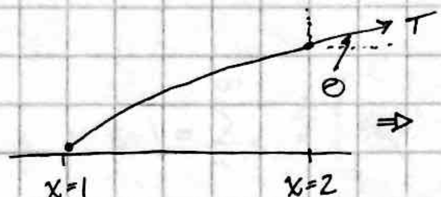


FUNCTION THAT IS ODD W/ RESPECT TO 0, EVEN W/ RESPECT TO ℓ. <u>4ℓ PERIODIC</u>. FORIER SERIES STILL HOLDS, JUST THAT EVERY ℓ, IS REPLACED BY 2ℓ.

$\boxed{10.7}$

TENSION = T , DENSITY = ρ

$u(x,t)$ = VERTICLE DISPLACEMENT



$$\Rightarrow \underbrace{\frac{d}{dt}\int_{x=1}^{x=2} \rho\, u_t(x,t)\, ds}_{\text{WILL EQUAL VERTICLE FORCES}}$$

VERTICLE FORCE:
$= T\sin\theta$

WE WILL APPROX AS:

$\cong \underbrace{T\tan\theta}$ (FOR θ SMALL)

↳ SLOPE OF THE LINE
$= T u_x(x,t)$

$$\frac{d}{dt}\int_{x=1}^{x=2} \rho u_t(x,t)\, ds = T u_x(x_2,t) - T u_x(x_1,t)$$

$$\Rightarrow \int_{x_1}^{x_2} \rho\, u_{tt}(s,t)\, ds = \int_{x_1}^{x_2} \rho = \int \frac{\partial}{\partial x} T u_x(s,t)\, ds + \int_{x_1}^{x_2} \rho F(s)\, ds$$

$u_{tt} - \alpha^2 u_{xx} = F(x,t)$; REASONABLE BC'S: $u(0,t) = 0$, $u(\ell,t) = 0$
$u(x,0) = f(x)$
$u_t'(x,0) = g(x)$

ALTERNATIVE BC'S:

$x = \ell$

1.) $u(\ell,t) = u_\ell(t)$

2.) $u_x(\ell,t) = u_\ell'(t)$

3.) $Tu_x + u(\ell,t)K = 0$

$$u_{tt} - \alpha^2 u_{xx} = 0$$

$$u(0,t) = 0, \quad u(\ell,t) = 0$$

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

SEPARATE VARIABLES:

TRY $u = X(x)T(t) \Rightarrow X(x)T'' = \alpha^2 X''(x)T \Rightarrow$

$$\frac{X''}{X} = \frac{T''}{\alpha^2 T} = -\lambda \; ; \quad X'' + \lambda X = 0 \; ; \quad T'' + \alpha^2 T = 0$$

$$\boxed{\begin{array}{l} X'' + \lambda X = 0 \\ X(0)=0, \; X(\ell)=0 \end{array}} \longleftarrow \text{E.V. PROBLEM}$$

$$X_n = \sin\left(\frac{n\pi}{\ell}x\right), \quad T_n = c_n\cos\left(\frac{n\pi}{\ell}\alpha t\right) + d_n\sin\left(\frac{n\pi\alpha}{\ell}t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_n\cos\left(\frac{n\pi\alpha}{\ell}t\right) + d_n\sin\left(\frac{n\pi\alpha}{\ell}t\right)\right)\sin\left(\frac{n\pi}{\ell}x\right)$$

SOLUTION W/OUT IC'S

IC's: $f(x)$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{\ell}x\right), \quad 0 \leq x \leq \ell$$

$$C_n = \frac{2}{L}\int_0^{\ell} f(x) \sin\left(\frac{n\pi}{\ell}x\right) dx$$

$$g(x) =$$

$$\frac{n\pi\alpha}{\ell} d_n = \frac{2}{\ell}\int_0^{L} g(x) \sin\left(\frac{n\pi}{\ell}x\right) dx$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{n\pi}{\ell}x\right) dx \cos\left(\frac{n\pi\alpha}{\ell}t\right) \sin\left(\frac{n\pi}{\ell}x\right) + \lambda_n +$$

$$\sum_{n=1}^{\infty} \frac{2}{n\pi\alpha}\int_0^{L} g(x) \sin\left(\frac{n\pi}{\ell}x\right) dx \sin\left(\frac{n\pi}{\ell}\alpha t\right) \sin\left(\frac{n\pi}{\ell}x\right)$$

FEB 8

$$\left[\begin{array}{l}\text{PDE}\\ \text{BC}\\ \text{IC}\end{array}\right.$$

SAME PATTERN FOR DIFFUSION/WAVE

SEPARATE VARIABLES

TWO-POINT BVP

EIGENFUNCTIONS/VALUES

GENERAL SOLUTION, PDE BC

IC: FORIER EXPANSION

FORIER EXPANSION:

$$\left.\begin{array}{l} f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{array}\right\} \text{2L PERIODIC}$$

$f(x) = (0,L)$	$f(x)$ ODD, COSINES CANCEL
$f(x) =$	EVEN, SINES CANCEL

IF DERIVATIVES VANISH @ ENDPOINTS WE WANT COSINES!

$$f(x) = \sum_{n=1}^{\infty} C_n X_n(x)$$

$\{X_n(x)\}$ ARE ORTHAGONAL:

EXAMPLES:

* WILL NOT NEED TO CALCULATE THE a_n's / b_n's

DIFFUSION EQUATION:

$$u_t - \alpha^2 u_{xx} = F(x,t) \leftarrow \text{MOST GENERAL}$$

$u_x(0,t) = u_0(t)$
$u_x(l,t) = u_l(t)$
$u(x,0) = f(x)$

1.) HOMOGENEOUS BOUNDARY CONDITIONS, USE LINEARITY
- SOLVE W/ HOMOGENEOUS BC'S, SUBTRACT PARTICULAR

$$U_t(x,t) - \alpha^2 U_{xx} = F(x,t) - v_t + \alpha^2 v_{xx}$$

$u(x,t) = U(x,t) + v(x,t)$

HOMOGENEOUS

$$\boxed{\begin{aligned} V_x(0,t) &= u_0(t) \\ V_x(l,t) &= u_l(t) \end{aligned} \Bigg\} \text{ SEPARATE B.V. PROBLEM}}$$

ANY FUNCTION V THAT SATISFYIES THIS, SOLVES THE B.V.P W/ NON-HOMO B.C.S

$$\left.\begin{aligned} V &= C_1 x + C_2 x^2 \\ V_x &= C_1 + C_2 2x \end{aligned}\right\} \text{IN BOOK}$$

IDEA IS WE CAN ALWAYS GET DOWN TO A PROBLEM W/ HOMO B.C'S.

$$u_t - \alpha^2 u_{xx} = F^*(x,t)$$
$$u_x(0,t) = 0 \quad u_x(\ell,t) = 0$$
$$u(x,0) = f^*(x)$$

HOMOGENEOUS 'PART'

2.) $f^* = 0$, $V(x,t)$ ⇐ SOLUTION w/ ~~[illegible]~~ $F(x,t) = 0$
GIVES V

3.) HOMOGENEOUS I.C. $f^*(x) = 0$ FOR $W(x,t)$

$\boxed{u = V + W}$ $\boxed{\text{LINEARITY}}$

#2. (STEP 2 EXPLAINED)

$$V = XT$$

$$\boxed{\begin{array}{l} X'' + \lambda X = 0 \\ X'(0) = 0,\ X'(\ell) = 0 \end{array}} \quad T' + \alpha^2 \lambda T = 0$$

MOST IMPORTANT PART

$$V(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

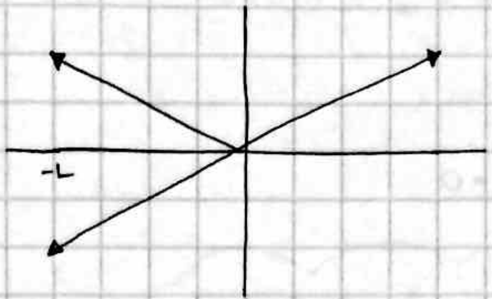
#3.) SOLVE FOR W, LOOK IN BOOK FOR SOLVING w/ F^* DEPENDENT ON x,t

$F^*(x,t)$

$$\text{TRY} \quad W(x,t) = \sum_{n=1}^{\infty} w_n(t) X_n(x)$$

FEBRUARY 13, 2012

1.)



2.) $u_t - u_{xx}\alpha^2 = 0 \qquad u(0,t) = 0, \quad u(\ell,t) = 0, \quad u(x,0) = 1$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$u(x,0) = 1 \Rightarrow u(x,0) = \sum c_n \sin\left(\frac{n\pi x}{L}\right) = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{L}x\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right)\right)$$

$$u(x,t) = \frac{4}{\pi}\left(e \ldots \sin\left(\frac{\pi}{\ell}x\right)\right.$$

$c_n = \frac{4}{\pi}$ N, ODD

$$= \sum \left(\frac{4}{\pi}\frac{1}{n}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

3.) $u_t - \alpha^2 u_{xx} = 0 \qquad u(0,t) = 0 \quad u(\ell,t) = 1 \quad u(x,0) = 0$

$$u(x,t) = w(x,t) + v(x) \qquad v(x) = \frac{x}{L}$$

$$w(0,t) = w(\ell,t) = 0 \qquad w(x,0) = -\frac{x}{L}$$

$$w(x,t) = \sum c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} = -\frac{x}{L}$$

$$= \frac{1}{L}\left(\frac{2\ell}{\pi}\left(\sin\left(\frac{\pi}{L}x\right) - \frac{\ell}{2}\sin\left(\frac{2\pi}{\ell}x\right)\right) \ldots\right.$$

$$u(x,t) = \frac{-2}{\pi}\left(e^{-\left(\frac{\pi\alpha}{L}\right)^2 t} \sin\left(\frac{\pi}{L}x\right) - \frac{1}{2} e^{-\left(\frac{2\pi\alpha}{\ell}\right)^2 t} \sin\left(\frac{2\pi}{\ell}x\right)\right)$$

$$u(x,t) = \frac{x}{L} - \frac{2}{\pi}\sum \frac{(-1)^n}{n} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\left(\frac{n\pi}{\ell}x\right)$$

4) $u_t - \alpha^2 u_{xx} = 1 \quad u(0,t) = 0, \; u(\ell,t) = 0 \quad u(x,0) = 0$

TRY:

$$u(x,t) = U(x,t) + v(x)$$

$$-\alpha^2 v''(x) = 1 \quad v(0) = v(\ell) = 0$$

$$F(F(x)) = \frac{-x^2}{2\alpha^2}$$

$$v(x) = c_1 + c_2 x - \frac{x^2}{2\alpha^2} \Rightarrow v(x) = \frac{\ell x - x^2}{2\alpha^2}$$

$$U(x,t) = \sum c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

$$U(x,0) = \sum c_n \sin\left(\frac{n\pi x}{L}\right) = \frac{x^2 - \ell x}{2\alpha^2}$$

$$= \frac{-1}{2\alpha^2} \frac{2\ell}{\pi^3}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3^3}\sin\left(\frac{3\pi x}{L}\right)\right.$$

$$U(x,t) = \frac{-4L^2}{\alpha^2\pi^3}\left(e^{-\left(\frac{\pi x}{L}\right)^2 t} \sin\left(\frac{\pi x}{L}\right)\right.$$

$$= \frac{-4L^2}{\alpha^2\pi^3} \sum_{n=1,3,5} \frac{1}{n^3} e^{-\left(\frac{n\pi x}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = \frac{\ell x - x^2}{2\alpha^2} +$$

ALSO; WHEN $u(0,t) = 0 = u(\ell,t) = 0 = u(x,0) = 0$

$$u(x,t) = \sum u_n(t) \sin\left(\frac{n\pi x}{L}\right) \; . . . \; \text{BC's}$$

INTO PDE:

$$\sum_{n=1}^{\infty} \left(u_n'(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 u_n(t)\right) \sin\left(\frac{n\pi x}{L}\right) = 1, \quad 0<x<\ell$$

$$= f_1$$

#4.) CONT.

$$u_n'(t) + \left(\frac{\pi n \alpha}{\ell}\right)^2 u_n(t) = \underbrace{\begin{matrix}\frac{4}{\pi n} & (n \text{ ODD}) \\ 0 & (n \text{ EVEN})\end{matrix}}_{f_1(x)}$$

$$u_n(t) = \boxed{\frac{4}{\pi n}\left(\frac{\ell}{n\pi\alpha}\right)^2\left(1 - e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\right) \quad \underline{\underline{(n \text{ ODD})}}}$$

WHERE $u(x,t) = \sum u_n(t) \sin\left(\frac{n\pi x}{L}\right)$

FEBRUARY 15, 2012

* SEQUENCES

A SEQUENCE IS A FUNCTION $\mathbb{N} \longrightarrow \mathbb{R}$

EX.

$$a_n = \frac{1}{n} \Rightarrow \left\{ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right\}$$

$$\lim (a_n) = \lim \left\{ \frac{1}{n} \right\}$$

DEF.

$\lim\limits_{n \to \infty} a_n = a$, FOR EACH $c > 0$, THEN THERE IS N:

$\forall \ n \gg N \quad |a_n - a| < c$

* CONTINUITY:

f IS CONTINUOUS @ a, $\lim\limits_{n \to \infty} = a \quad (a_n \Rightarrow a)$

$$\lim_{n \to \infty} f(a_n) = f(a)$$

* FACT:

$$(1+h)^n \geq 1 + hn \qquad n > 0, \ n \in \mathbb{N}$$

SEQUENCES, CONT.

* CAUCHY TEST:

SEQUENCE $\{a_n\}$ IS CONVERGENT $\iff \forall \varepsilon < 0$, THERE IS AN N $m, n \geq N \Rightarrow |a_n - a_m| < \varepsilon$.

LET $\{a_n\}$ BE A SEQUENCE, CONSTRUCT ANOTHER SEQUENCE $\{S_n\}$ BY $S_n = a_1 + a_2 + a_3 \dots = \sum_{m=1}^{n} a_n$

SEQUENCE OF "PARTIAL SUMS" $\iff S_n \to S$

EX.

LET $|p| < 1$.

$a_n = p^n \rightarrow$ CONVERGENT TO 0

S_n (SERIES) $= 1 + p + p^2 + p^3$

$S_n - pS_n = -(p + p^2 + p^3 + p^n + p^{n+1})$

$= 1 - p^{n+1}$

$$S_n = \frac{1 - p^{n+1}}{1 - p} \Rightarrow \lim_{n \to \infty} S_n = \frac{1}{1 - p}$$

$$\sum_{n=0}^{\infty} p^n = \dots \frac{1}{1 - p}, \quad |p| < 1$$

GEOMETRIC SERIES

ALTERNATING SERIES :

$$\sum_{n=0}^{\infty} (-1)^n a_n , \quad a_n > 0$$

ALTERNATING SERIES TEST . . .

THE INTEGRAL TEST:

$$\sum_{n=1}^{\infty} f(n) \text{ CONVERGES } \Longleftrightarrow \int_1^{\infty} f(x)\,dx \text{ CONVERGES}$$

* ABSOLUTE CONVERGENCE: $\sum a_n$ IS ABSOLUTELY CONVERGENT IF THE SERIES $\sum |a_n|$ CONVERGES

* CONDITIONAL CONVERGENT: $\sum a_n$ IS CONDITIONAL CONVERGENT IF $\sum a_n$ CONVERGES BUT $\sum |a_n|$ DOES NOT.

* COMPARISON TEST : IF $\sum a_n$ AND $\sum b_n$ ARE SERIES WITH $|a_n| \leq C b_n$, $n \geq N$ AND $\sum b_n$ CONVERGE, THEN $\sum a_n$ IS ABSOLUTELY CONVERGENT

* KNOW THE RATIO TEST, "LIMIT RATIO TEST", ROOT TEST. POINT-WISE CONVERGENCE THEORUM, UNIFORM CONVERGENCE.

FEBRUARY 22, 2012

$$\left.\begin{array}{l} y' = y \\ y(0) = 1 \end{array}\right\}$$

TRY $y(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_{n-1} x^{n-1} + a_n x^n$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + N a_n x^{n-1}$$

MATCHING COEFFICIENTS :

$$\underbrace{\begin{array}{l} a_1 = a_0 \\ a_2 = \frac{1}{2} a_1 \\ a_3 = \frac{1}{3} a_2 \\ a_n = \frac{1}{n} a_{n-1} \end{array}}$$

$$y(x) = a_0\left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots\right)$$

$$y(0) = a_0 = 1$$

<u>POWER</u> <u>SERIES:</u>

$$\sum a_n (x - x_0)^n = \ldots \quad \text{(SECTION 5.1)}$$

LET $\{f(x\}$ BE AN INTEGRABLE FUNCTION ON $[a, b]$.
ASSUME $f_n \to f$ <u>UNIFORMLY</u> ON $[a, b]$. FOR EVERY
$\epsilon > 0$, THEN N:

$$N \Rightarrow N \Rightarrow |(f_n(x) - f(x)| < \varepsilon \quad \text{ALL } x$$

SECTION 5.1 : 1-15, 19-21
5.2 : 1-5

* RADIUS OF CONVERGENCE $\sum_{n=1}^{\infty} a_n(x-x_0)$

- $x = x_0$
- ALL x
- $(x - x_0) < R$ ⟵ WHERE R = RADIUS OF CONVERGENCE

EX.

$$\sum (n+1) x^n = 1 + 2x + 3x \ldots +$$

RATIO TEST:

$$\frac{(n+2)|x|^{n+1}}{(n+1)|x|^n} \Rightarrow \frac{n+2}{(n+1)}(|x|)$$

$$\lim_{n\to\infty}\left(\frac{n+2}{n+1}|x|\right) = |x|$$, SO, $|x| < 1$ THE IT CONVERGES

$R = 1$

n^{TH} TERM MUST → TO ZERO FOR SERIES TO CONVERGE

INSIDE RADIUS OF CONVERGENCE, SERIES CONVERGES UNIFORMLY AND ABSOLUTELY

LET $n-1 = m$

$$\sum (n+1)n x^{n-1} = ((m+1)+1)(m+1) x^m \to (m+2)(m+1)x^m$$

$$\sum (n+2)(n+1)x^n \Rightarrow \lim\left(\frac{(n+3)(n+2)}{(n+2)(n+1)}\right) = 1 \quad R = 1$$

$$\int_0^x f(t)\,dt = \sum_{n=0} x^{n+1} \Rightarrow \sum_{n=1} x^n = \frac{1}{1-x} - 1 = \boxed{\frac{x}{1-x}}$$

$$e = \sum_{n=0} \frac{1}{n!} x^n \Rightarrow \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{(n+1)} \quad R = \infty$$

FIND SOLUTIONS TO :

$$y'' + y = 0$$

TRY $y = \sum_{n=0} a_n x^n \Rightarrow y' = \sum_{n=1} a_n n x^{(n-1)} \Rightarrow y'' = \sum a_n (n)(n-1) x^{n-2}$

NEED TO COMPARE? → SHIFT INDEX

$$y'' + y = \sum\left((n+2)(n+1)a_{n+2} + a_n\right) x^n = 0$$

$$(n+2)(n+1)a_{n+2} + a_n = 0, \quad n \geq 0$$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}, \quad n \geq 0$$

$$y(x) = \underbrace{a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\right)}_{\cos}, \quad \underbrace{a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right)}_{\sin}$$

2/27/2012

#9. $(1+x^2)y'' - 4xy' + 6y = 0$

TRY: $y = \sum_{n=0}^{\infty} a_n x^n$ SO, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$ AND

$$y'' = \underbrace{\sum_{n=0}^{\infty}}_{n=0,1,2} n(n-1) a_n x^{n-2}$$

WE HAVE THAT:

$$xy' = \sum_{n=0,1}^{\infty} n a_n x^{\cancel{n-1}} = xy' = \sum_{n=0,1}^{\infty} n a_n x^n$$

$$y'' = \sum (n+2)(n+1) a_n x^n \quad (\text{INDEX SHIFT OF } n+2)$$

ODE:

$$= \sum_{n=0}^{\infty} \Big((n+2)(n+1)a_{n+2} + a_n(n(n+1) + n + 1) \Big) x^n$$

$$= \sum_{n=0}^{\infty} \Big((n+2)(n+1)a_{n+2} + a_n(n(n-1) - 4n + 6) \Big) x^n$$

$$(n+2)(n+1)a_{n+2} + a_n((n-2)(n-3)) = 0, \quad n \geq 0$$

$$\boxed{a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)} a_n \quad n \geq 0}$$

RECCURSION FORMULA (MOST IMPORTANT STEP)

$a_0 = (= y(0))$ $\quad a_1 = (= y'(0))$ $\quad a_2 = \frac{-(-2)(-3)}{1+2} a_0$

$a_3 = \frac{(-1)(-2)}{2 \cdot 3} a_1$

$$y(x) = a_0(1 - 3x^2) + a_1\left(x - \tfrac{1}{3}x^3\right)$$

2/27/2012

$(4-x^2)y'' + 2y = 0$ TRY, $\sum_{n=0}^{\infty} a_n x^n$ SO $y' = \sum n a_n x^{n-1}$

AND $x^2 y'' = \sum n(n-1) a_n x^n \Rightarrow y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n$

ODE:

$$\sum \left(4(n+2)(n+1)a_{n+2} - a_n\left(\underbrace{n(n-1)-2}_{n^2-n-2 \,=\, (n-2)(n+1)}\right)\right) x^n$$

$$\boxed{a_{n+2} = \frac{(n-2)}{4(n+2)} a_n, \ n \geq 0}$$

$a_0 = \ldots$ $\qquad a_1 = \ldots$

$a_2 = \frac{-1}{4} a_0$ $\qquad a_3 = \frac{-1}{4(3)} a_1$

$a_4 = 0$ $\qquad a_5 = \frac{1}{4(5)} a_3 = \frac{-1}{4^2 \cdot 3 \cdot 5}$

$a_6 = 0$ $\qquad a_7 = \frac{3}{4 \cdot 5} a_5$

$y(x) = a_0\left(1 - \frac{1}{4}x^2\right) + a_1\left(x - \frac{1}{3 \cdot 4}x^3 - \frac{1}{3 \cdot 5 \cdot 4^2} x^5\right.$

RADIUS OF CONVERGENCE $= 2$ FROM RECURSION FORMULA.

3/2/2012

TEST # SECTIONS 5.2, 5.3

1.) FOR THE GIVEN DIFF EQ. DETERMINE THE LOWER BOUND ON RADIUS OF CONVERGENCE

2.) FIND RECURRANCE RELATION

3.) USE #2 TO FIND 2 INDEPENDENT SOLUTIONS

4.) FIND RADIUS OF CONVERGENCE

FIND THESE IN 5.2/5.3 AND WORKOUT

EX. $y'' + x^2 y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y'' = \sum_{n=0,1,2} n(n-1) a_n x^{n-2}$$

$$y'' = \sum (n+2)(n+1) a_{n+2} x^n$$

$$x^2 y = \sum_{n=1}^{\infty} a_n x^{n+2} = \sum_{\underline{\underline{n=2}}} a_{n-2} x^n$$

$$\text{ODE} \Rightarrow \quad 2a_2 + 6a_3 x + \sum_{n=2} \left((n+2)(n+1) a_{n+2} + a_{n-2}\right) x^n = 0$$

$$\boxed{a_{n+1} = \frac{-a_{n+2}}{(n+2)(n+1)}, \quad n \geq 2 \quad \& \quad a_2 = 0, \; a_3 = 0}$$

$$y(x) = a_0\left(1 - \frac{1}{2 \cdot 4} x^4 + \frac{1}{(3 \cdot 4)(7 \cdot 8)} x^8 + \frac{1}{(3 \cdot 4)(7 \cdot 8)(11 \cdot 12)} x^{12} + \ldots\right.$$
$$a_1\left(x - \frac{1}{4 \cdot 5} x^5 + \frac{1}{(4 \cdot 5)(8 \cdot 9)} x^9 \ldots\right)$$

3/2/2012

#9.)

$$(1+x^2)y'' - 4xy' + 6y = 0, \quad x_0 = 0$$

R=1, LOWER BOUND

$$y = \sum_{n=0} a_n x_n$$

$$y' = \sum a_n n(x^{n-1})$$

$$y'' = \sum_{n=0,1,2} n(n-1)x^{n-2}$$

POWER SERIES REVIEW NOTES:

* LIMIT COMPARISON TEST:

SUPPOSE THAT $\sum a_n$ AND $\sum b_n$ ARE SERIES W/ POSITIVE TERMS. IF $\lim_{n \to \infty} \frac{a_n}{b_n} = C$ WHERE C IS FINITE AND $C > 0$, THEN EITHER BOTH THE SERIES CONVERGE OR DIVERGE

* ALTERNATING SERIES TEST:

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \ldots$ $b_n > 0$ SATISFIES

I.) $b_{n+1} \leq b_n$ FOR ALL $n \in \mathbb{R}$

II.) $\lim_{n \to \infty} b_n = 0$

THE SERIES IS CONVERGENT

* RATIO TEST:

IF $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ THEN THE SERIES $\sum_{n=1}^{\infty} a_n$ IS ABSOLUTELY CONVERGENT

IF $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ OR IF $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, THEN THE SERIES $\sum_{n=1}^{\infty} a_n$ IS DIVERGENT

IF $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, THE RATIO TEST IS INCONCLUSIVE

* ROOT TEST:

IF $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$, THEN THE SERIES $\sum_{n=1}^{\infty} a_n$ IS ABSOLUTELY CONVERGENT

IF $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1$, THEN THE SERIES IS DIVERGENT

IF $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$, THE ROOT TEST IS INCONCLUSIVE

* THE GEOMETRIC SERIES:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$ IS CONVERGENT

IF $|r| < 1$ AND ITS SUM IS $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ $|r|<1$

IF $|r| \geq 1$ THE GEOMETRIC SERIES IS DIVERGENT.

* RADIUS OF CONVERGENCE FOR TAYLOR SERIES:

"LOWER BOUND" OF CONVERGENCE IS PRECISELY WHERE P HAS ZERO'S. THIS IS THE MINIMUM SET OF CONVERGING DOMAIN FOR THE SERIES.

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5.4: 1-12

$$\frac{dy}{dx} = \frac{dy}{dt}$$

$x^2y'' + \alpha x y' + \beta y = 0$ EULER TYPE ODE

TRY: $y = x^r$

SIMILAR TO:

SO $\left(r(r-1) + \alpha r + \beta\right)x^r$

$ay'' + by' + cy = 0$, TRY $y = e^{rt}$,

$(ar^2 + br + c)\cancel{e^{rt}} = 0$

CHANGE OF VARIABLE (EULER): $x = e^t$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt} \Rightarrow \frac{d^2y}{(dx)^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right)$$

$$= \frac{1}{x^2}\frac{dy}{dt^2} - \frac{1}{x}\frac{dy}{dt}$$

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + \alpha\frac{dy}{dt} + \beta y = 0 \Rightarrow \frac{d^2y}{dt^2} + (\alpha - 1)\frac{dy}{dt} + \beta y = 0$$

$y = e^{rt}$: $r^2 + (\alpha - 1)r + \beta = 0$

EX:

$x^2y'' + 4xy' + 2y = 0$, TRY $y = x^r$

$r(r-1) + 4r + 2 = 0$

$r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0 \quad r = -2, -1$

$$\boxed{y = c_1 x^{-2r} + c_2 x^{-r}}$$

EX: $x^2y'' + 3xy' + 4y = 0, \quad y = x^r$

$$r(r-1) - 3r + 4 = 0$$

$$(r-2)^2 = 0, \quad r = 2,2 \Rightarrow y = c_1 x^2 + c_2(\ln(x))x^2$$

EX: $x^2y'' - 2xy' + 6y = 0$

$$r(r-1) - 2r + 6 = 0 \Rightarrow r^2 - 3r + 6 = 0$$

$$r^2 - 3r + 9/4 + (6 - 9/4) = 0$$

$$(r^2 - 3/2)^2 + 15/4 = 0, \quad r = 3/2 \pm i \frac{\sqrt{15}}{2}$$

$$x^{3/2}\cos\left(\frac{\sqrt{15}}{2}\ln(x)\right), \quad x^{3/2}\sin\left(\frac{\sqrt{15}}{2}\ln(x)\right)$$

NOW, SUPPOSE: $x^r = x^{\lambda + \mu i} = x^{\lambda} x^{i\mu}$

$$x^{i\mu} = e^{it\mu} = \cos(t\mu) + i\sin(t\mu)$$

$$= \cos(\mu \ln(x)) + i\sin(\mu \ln(x))$$

$$e^{it} = x(t) + iy(t) \qquad \frac{d}{dt}e^{it} = ie^{it} \text{ AND } e^{i(0)} = 1$$

SO, $x'(t) + iy'(t) = i(x(t) + iy(t))$

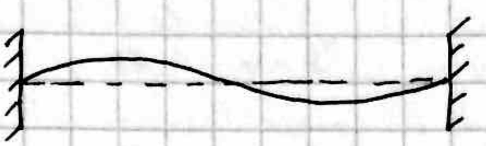
$$\begin{cases} x'(t) = -y(t) & x(0) = 1 \\ y'(t) = x(t) & y(0) = 0 \end{cases}$$

$x''(t) + x(t) = 0, \; x(0) = 1, \; x'(0) = 0$, SO $x(t) = \underline{\underline{\cos(t)}}$

$y(t) = -x'(t) = \underline{\underline{\sin(t)}}$

11.1: 7-10, 16

BOUNDARY VALUE PROBLEMS



$u(x,t)$ = UPWARD DISPLACEMENT

ROTATING AT FREQUENCY ω, $F(x,t)$ BECOMES $\rho\omega^2 u(x,t)$

$$\rho \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} T \frac{\partial u}{\partial x} = F(x,t)$$

STATIONARY SOLUTION FOR ROTATION:

$$\boxed{\begin{matrix} -Tu''(x) = \rho\omega^2 u(x) \\ u(0)=0 \quad u(\ell)=0 \end{matrix}} \Rightarrow u''(x) = \left(\frac{\rho}{T}\omega^2\right)u(x)$$

TWO-POINT B.V.P:

$$\boxed{\begin{matrix} -u''(x) + \lambda u(x) = 0 \\ u(0)=0 \quad u'(\ell)+u(\ell)=0 \end{matrix}}$$

$\lambda > 0$

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$y = y(x)$

BVP:

$$\boxed{\begin{aligned} -y'' + y &= f(x) \\ y(0) = \alpha&,\ y(\ell) = \beta \end{aligned}}$$

GENERAL SOLUTION:

1.) $f(x) = 0$,

$y = c_1 e^{x} + c_2 e^{-x}$ OR $\Rightarrow$ $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$

$\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

SO,

$x = 0$: $c_1 = \alpha$

$x = \ell$: $c_1 \cosh(x) + c_2 \sin(\ell) = \beta$

$$c_2 = \frac{\beta - \alpha \cosh(\ell)}{\sinh(\ell)}$$

$$\boxed{y(x) = \alpha \cosh(x) + \frac{\beta - \alpha \cosh(\ell)}{\sinh(\ell)} \sinh(x)}$$

2.) $-y''(x) = f(x)$

$y(0) = \alpha,\ y(\ell) = \beta$

$f(x) = 0 \Rightarrow y = c_1 + c_2 x$

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$\alpha = \beta = 0$

$$y'(x) = C_2 - \int f(x)\,dx$$

$$y(x) = C_1 + C_2 - \int_0^x \int_0^s f(y)\,dy\,ds \Rightarrow C_1 = 0 \quad y(x) = C_2 x - \int_0^x \int_0^s f(y)\,dy\,ds$$

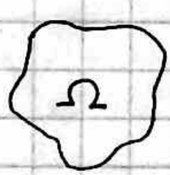
$$y(\ell) = C_2 \ell - \int_0^\ell \int_0^s f(y)\,dy\,ds \Rightarrow C_2 = \frac{1}{\ell}\int_0^\ell \int_0^s f(y)\,dy\,ds$$

$$y(x) = \frac{x}{\ell}\int_0^\ell \int_0^s f(y)\,dy - \int_0^x \int_0^s f(y)\,dy\,ds =$$  $$= \underbrace{\int_0^\ell G(x,y) = f(y)\,dy}$$

GREEN'S FUNCTION
FOR B.V.P.

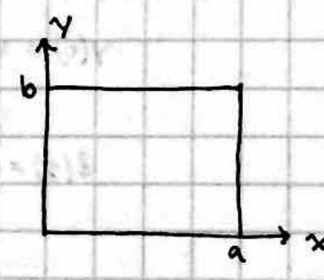
<u>10.8</u> 1-4 TWO-VARIABLE ANALOGUE:

$u = u(x,y)$: $-u_{xx} - u_{yy} = f(x,y)$, $(x,y) \in \Omega$ IN PLANE



Γ = BOUNDARY OF Ω

DIRICHLET PROBLEM



$$\Omega = [0,a] \times (0,b) \quad \left\{(x,y) : \begin{matrix} 0 \le x \le a \\ 0 \le y \le b \end{matrix}\right.$$

$u_{xx} + u_{yy} = 0$

$u(0,y) = u(a,y) = 0$

$u(x,0) = 0$, $u(x,b) = g(x)$

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TRY $= u(x,y) = X(x)\,Y(x)$

$$X''Y + XY'' = 0 \implies \frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

$$X'' - \lambda X = 0\,, \quad Y'' + \lambda Y = 0$$

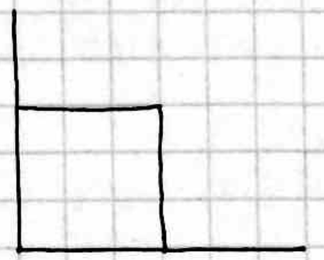
$$= \ldots \sum_{n=1}^{\infty} c_1 \sin\left(\frac{n\pi}{\alpha}x\right)\sinh\left(\frac{n\pi}{\alpha}y\right)$$

$$c_n \sin\left(\frac{n\pi}{a}\right) = \frac{2}{a}\int_0^a \gamma(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

LAPLACE'S EQUATION:

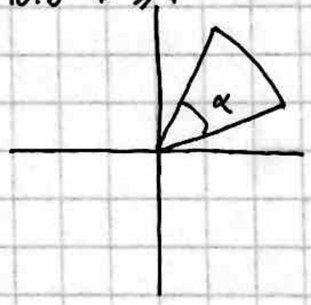
$$u_{xx} + u_{yy} = 0$$

CAN BE DEFINED FOR DIFFERENT REGIONS:



'OR'

10.8 # 3/4



$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r}u_{\theta\theta}$$

$$r^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r}u_{\theta\theta}\right)$$

$$= r^2 u_{rr} + r u_r + r u_{\theta\theta}$$

$$u(r,0) = u(r,\alpha) = 0 \qquad 0 \leq r \leq R$$

$$u(R,\theta) = f(\theta) \qquad 0 \leq \alpha \leq \Theta$$

TRY TO SEPARATE:

$$r^2 X''Y + rX'Y + XY'' = 0$$

EULER TYPE EQUATION

$$r^2\frac{X''}{X} + r\frac{X'}{X} = \frac{-Y''}{Y} = \lambda$$

$$\Rightarrow \quad r^2X'' + rX' - \lambda X = 0$$

$$\boxed{\begin{array}{c} Y'' + \lambda Y = 0 \\ Y(0) = Y(\alpha) = 0 \end{array}}$$

E.V.P

$$Y(\theta) = \sin\left(\frac{n\pi\theta}{\alpha}\right), \quad n \geq 1$$

EXPECT A EULER TYPE: $X = r^m \Rightarrow$ PLUGGING IN:

$$X = r^m \Rightarrow m(m-1) + m - \lambda = 0 \Rightarrow m^2 - \lambda = 0$$

$m = \pm \frac{n\pi}{\alpha} \Rightarrow$ SOLUTION FOR $X_n(r) = \cancel{r^{-n\pi/\alpha}}^{\,0} + c_1 r^{n\pi/\alpha}$

EXPECT:

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi\theta}{\alpha}\right)$$

* BOUNDARY CONDITIONS DETERMINE EVERYTHING *

WE NEED:

$$u(R,\theta) = \sum \underbrace{a_n R^{n\pi/\alpha}}_{} \sin\left(\frac{n\pi}{\alpha}\right) = f(\theta)$$

$\rightarrow$ FORIER SINE SERIES

BACK TO WAVE EQUATION (10.7)

* FINAL EXAM WILL HAVE ONLY PROBLEMS FROM 10.7/10.8 *

$$u_{tt} - \alpha^2 u_{xx} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(0,t) = u(\ell,t) = 0$$

$$u_t(x,0) = g(x)$$

$$u(x,0) = f(x)$$

TRY $u = X(x)T(t) \Rightarrow \frac{X''}{X} = \frac{T''}{T\alpha^2} = -\lambda$

$X'' + \lambda X = 0$ $\qquad T'' + \alpha^2 \lambda T = 0$

$X(0) = X(\ell) = 0$

$$T = c_1 \cos\left(\frac{n\pi\alpha}{\ell} t\right) + c_2 \sin\left(\frac{n\pi\alpha}{\ell} t\right)$$

$$\underbrace{u(x,t) = \sum_{n=1}^{\infty} \left\{a_n \cos\left(\frac{n\pi\alpha}{\ell} t\right) + b_n \sin\left(\frac{n\pi\alpha}{\ell} t\right)\right\} \sin\left(\frac{n\pi}{\ell} x\right)}_{\text{SOLVES PDE, AND B.C'S}}$$

I.C's $\quad u(x,0) = \underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{\ell} x\right)}_{\text{FORIER S.S}} = f(x)$

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi\alpha}{\ell}\right) \sin\left(\frac{n\pi}{\ell} x\right) = g(x)$$

* EXPECT DIFFERENT BOUNDARY CONDITIONS W/ WAVE, SIMILAR TO WORKED EXAMPLE W/ WEDGE LAPLACE *

* VALUE OF DERIVATIVE = 0 TYPE PROBLEMS

SECTION 5.4 READING NOTES:

* EULER EQUATIONS:

- HAVE FORM: $P(x)y'' + Q(x)y' + R(x)y = 0$, A COMMON EQUATION OF THIS TYPE IS THE CAUCHY-EULER EQUATION:

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0$$ WHERE α, β ARE BOTH REAL CONSTANTS.

* REAL, DISTINCT ROOTS:

- IN THE CASE OF REAL DISTINCT ROOTS, $W(x^{r_1}, x^{r_2}) = (r_2 - r_1)x^{r_1+r_2-1}$ WHICH IS NON-VANISHING FOR $r_1 \neq r_2$. THEREFOR, FOR REAL, DISTINCT ROOTS;

$$\boxed{y = c_1 x^{r_1} + c_2 x^{r_2}}$$

* REPEATED ROOTS (EQUAL ROOTS)

$$\boxed{y = c_1 x^{r_1} + c_2 \ln(x) x^{r_1}}$$

WAVE EQUATION NOTES:

* ELASTIC STRING WITH NON-ZERO INITIAL DISPLACEMENT.

$$a^2 u_{xx} = u_{tt} \quad 0 \leq x \leq L \ , \ t > 0$$

- BOUNDARY CONDITIONS:

$$u(0,t) = 0 \quad u(L,t) = 0 \quad t \geq 0$$

- INITIAL CONDITIONS:

$$u(x,0) = f(x) \quad u_t(x,0) = 0 \quad 0 \leq x \leq L$$

WHERE $f(x)$ IS A GIVEN FUNCTION DESCRIBING THE CONFIGURATION OF THE STRING AT $t = 0$.

SOLUTION:

ASSUME $u(x,t) = X(x)T(t)$ $\Rightarrow$ INTO PDE $\Rightarrow$ $\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda$

SO WE GET: $\left.\begin{array}{l} X'' + \lambda X = 0 \\ T'' + a^2\lambda T = 0 \end{array}\right\}$ $X(0) = 0, \ X(\ell) = 0, \ T'(0) = 0$

$\lambda = \frac{n^2\pi^2}{L^2}$ $\Rightarrow$ $T'' + \frac{n^2\pi^2 a^2}{L} T = 0$, AN ODE, SO $T(t) = K_1 \cos\left(\frac{n\pi a t}{L}\right) + K_2 \sin\left(\frac{n\pi a t}{L}\right)$

BUT